

# *STERA 3D*

*Structural Earthquake Response Aalysis 3D*

**Technical Manual**

**Version 7.4**

Dr. Taiki SAITO

TOYOHOSHI UNIVERSITY OF TECHNOLOGY (TUT), JAPAN

## UPDATE HISTORY

2016/8/28	STERA_3D Technical Manual Ver.5.2 is uploaded. Definition of External Spring is extended in three directions.
2016/10/23	STERA_3D Technical Manual Ver.5.3 is uploaded. “5.5 Modal analysis”, “7.4 Calculation of ground displacement” are added.
2016/11/26	STERA_3D Technical Manual Ver.5.4 is uploaded. Formulation of initial stiffness of nonlinear spring is fixed (Eqs. (3-1-34), (3-1-51))
2017/01/18	STERA_3D Technical Manual Ver.5.5 is uploaded. “5.5 Modal analysis” is modified including participation factor, effective mass, etc.
2017/03/20	STERA_3D Technical Manual Ver.5.6 is uploaded. “7.4 Calculation of ground displacement” is modified changing band-pass filter.
2017/10/08	STERA_3D Technical Manual Ver.5.7 is uploaded. Ground springs are added.
2017/10/24	STERA_3D Technical Manual Ver.5.8 is uploaded. “4.6 Mass matrix corresponding to independent degrees of freedom” is added.
2019/02/12	STERA_3D Technical Manual Ver.6.0 is uploaded.
2019/05/20	Radiation damping for ground springs is added.
2019/07/25	External force by Wind is added.
2019/10/08	Buckling hysteresis of a brace is added.
2020/03/16	Pile foundation is included for ground springs. Air spring is added for an external spring.
2021/10/10	STERA_3D Technical Manual Ver.7.0 is uploaded. For RC column and RC wall, the nonlinear bending springs independent in x and y directions are introduced. For Steel beam, the nonlinear shear spring for hysteresis damper is introduced. Damage indices of members are introduced.
2022/08/22	STERA_3D Technical Manual Ver.7.1 is uploaded. The model of the direct input wall is changed to be the lumped mass model. For external springs, models of the base plate and the pendulum spring are introduced.
2022/12/14	STERA_3D Technical Manual Ver.7.2 is uploaded. For base isolation elements, FPB (Friction Pendulum Bearing) is introduced.
2023/03/13	STERA_3D Technical Manual Ver.7.3 is uploaded. The formula of compression strength of Masonry element is changed.
2023/06/06	STERA_3D Technical Manual Ver.7.4 is uploaded. Viscoelastic damper is added to the passive damper and the shear spring of steel beams.

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## 1. Basic Condition

### 1.1 Coordinate

#### (1) Global Coordinate

The global coordinate is defined as the left-hand coordinate as shown in Figure 1-1-1.

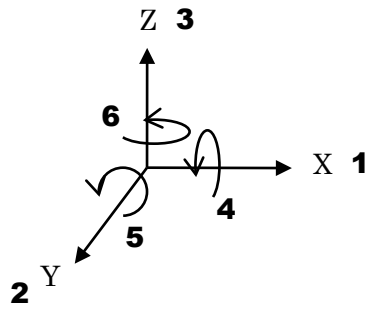


Figure 1-1-1 Global coordinate

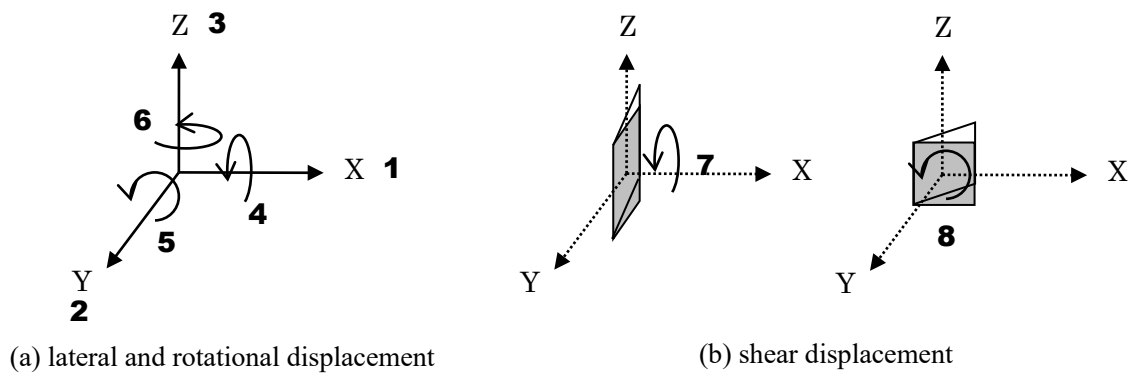


Figure 1-1-1 Global coordinate

## (2) Local Coordinate

The local coordinate is defined for each element. The displacement freedoms and force freedoms are named with subscripts indicating the coordinate direction and node name. For example, the local coordinate of a beam element in Figure 1-2 is defined to have its x-axis in the same direction of the element axis. Also the displacement and force freedoms of a beam element are expressed as shown in Figure 1-1-2.

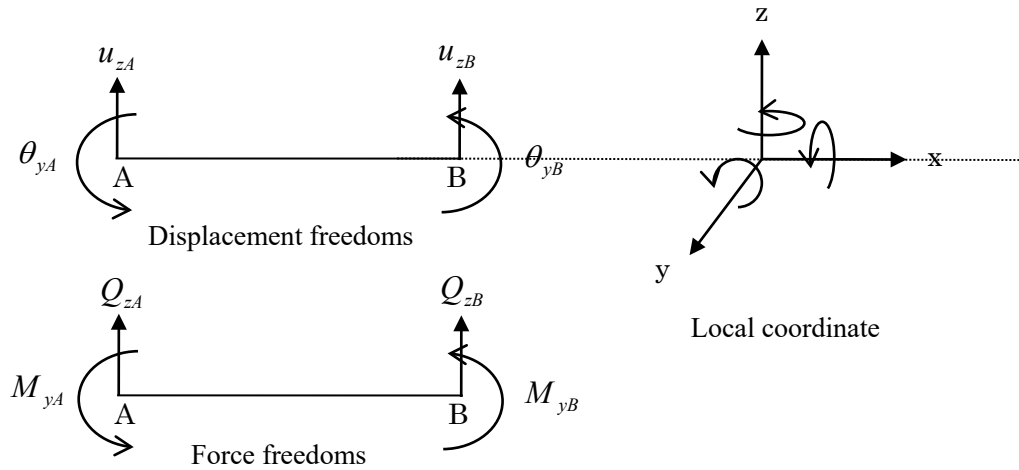


Figure 1-1-2 Local coordinate of a beam element

## 2. Constitutive Equation of Elements

### 2.1 Beam

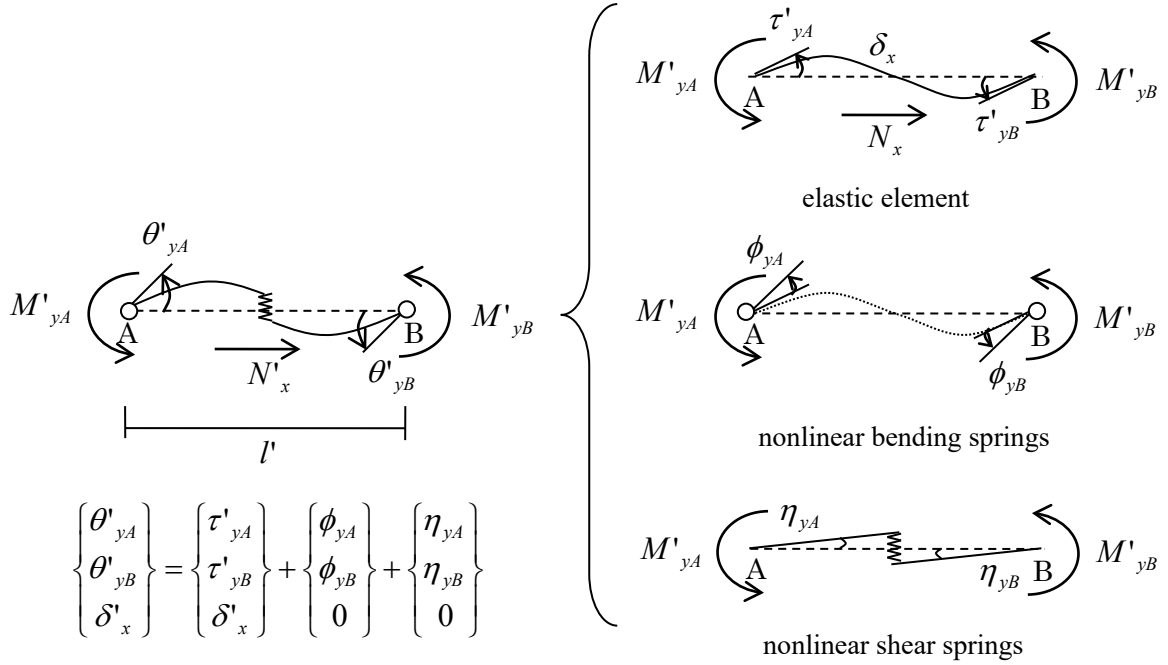


Figure 2-1-1 Element model for beam

#### Force-displacement relationship for elastic element

The relationship between the displacement vector and force vector of the elastic element in Figure 2-1-1 is expressed as follows:

$$\begin{Bmatrix} \tau'_{yA} \\ \tau'_{yB} \\ \delta'_x \end{Bmatrix} = \begin{bmatrix} \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ -\frac{l'}{6EI_y} & \frac{l'}{3EI_y} & 0 \\ 0 & 0 & \frac{l'}{EA} \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ N'_x \end{Bmatrix} \quad (2-1-1)$$

where,  $E$ ,  $I_y$ ,  $A$  and  $l'$  are the modulus of elasticity, the moment of inertia of the cross-sectional area along y-axis, the cross-sectional area and the length of the element. The rotational displacement vector of the nonlinear bending springs is,

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{yB} \end{Bmatrix} = \begin{bmatrix} f_{yA} & 0 \\ 0 & f_{yB} \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \end{Bmatrix} \quad (2-1-2)$$

where,  $f_{yA}$  and  $f_{yB}$  are the flexural stiffness of nonlinear bending springs at both ends of the element.

The force-deformation relationship of shear spring is

$$Q_z = k_z s_z \quad \text{or} \quad s_z = (1/k_z) Q_z$$

From the relationship between shear force and moment,

$$Q_x = [1/l' \quad 1/l'] \begin{Bmatrix} M_{yA} \\ M_{yB} \end{Bmatrix}$$

The end rotational displacement due to shear deformation is obtained as,

$$\begin{Bmatrix} \eta_{yA} \\ \eta_{yB} \end{Bmatrix} = \begin{bmatrix} 1/l' \\ 1/l' \end{bmatrix} s_z = \begin{bmatrix} 1/l' \\ 1/l' \end{bmatrix} (1/k_z) Q_z = \begin{bmatrix} 1/l' \\ 1/l' \end{bmatrix} (1/k_z) [1/l' \quad 1/l'] \begin{Bmatrix} M_{yA} \\ M_{yB} \end{Bmatrix} = \begin{bmatrix} \frac{1}{k_{sz} l'^2} & \frac{1}{k_{sz} l'^2} \\ \frac{1}{k_{sz} l'^2} & \frac{1}{k_{sz} l'^2} \end{bmatrix} \begin{Bmatrix} M_{yA} \\ M_{yB} \end{Bmatrix} \quad (2-1-3)$$

where,  $k_{sz}$  is the shear stiffness of the nonlinear shear spring. Then, the displacement vector of the beam element is obtained as the sum of the above three displacement vectors.

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta'_x \end{Bmatrix} = \begin{Bmatrix} \tau'_{yA} \\ \tau'_{yB} \\ \delta'_x \end{Bmatrix} + \begin{Bmatrix} \phi_{yA} \\ \phi_{yB} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \eta_{yA} \\ \eta_{yB} \\ 0 \end{Bmatrix} = [f_B] \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ N'_x \end{Bmatrix} \quad (2-1-4)$$

where,

$$[f_B] = \begin{bmatrix} f_{yA} + \frac{l'}{3EI_y} + \frac{1}{k_{sz} l'^2} & -\frac{l'}{6EI_y} + \frac{1}{k_{sz} l'^2} & 0 \\ & f_{yB} + \frac{l'}{3EI_y} + \frac{1}{k_{sz} l'^2} & 0 \\ \text{sym.} & & \frac{l'}{EA} \end{bmatrix} \quad (2-1-5)$$

$[f_B]$  is the flexural stiffness matrix of the beam element. By taking the inverse matrix of  $[f_B]$ , the constitutive equation of the beam element is obtained as,

$$\begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ N'_x \end{Bmatrix} = [f_B]^{-1} \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta'_x \end{Bmatrix} = [k_B] \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta'_x \end{Bmatrix} \quad (2-1-6)$$

where,  $[k_B]$  is the stiffness matrix of the beam element.

### Including rigid parts and node movement

Including rigid parts and node movement as shown in Figure 2-1-2, the rotational displacement vector is,

$$\begin{aligned} \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \end{Bmatrix} &= \begin{Bmatrix} \theta_{yA} - \tau \\ \theta_{yB} - \tau \end{Bmatrix}, \quad \tau = \frac{(u_{zB} - \lambda_B l' \theta_{yB}) - (u_{zA} + \lambda_A l' \theta_{yA})}{l'} \\ &= \begin{Bmatrix} \theta_{yA} + \frac{1}{l'} u_{zA} + \lambda_A \theta_{yA} - \frac{1}{l'} u_{zB} + \lambda_B \theta_{yB} \\ \theta_{yB} + \frac{1}{l'} u_{zA} + \lambda_A \theta_{yA} - \frac{1}{l'} u_{zB} + \lambda_B \theta_{yB} \end{Bmatrix} = \begin{bmatrix} \frac{1}{l'} & -\frac{1}{l'} & 1 + \lambda_A & \lambda_B \\ \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1 + \lambda_B \end{bmatrix} \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \end{Bmatrix} \quad (2-1-7) \end{aligned}$$

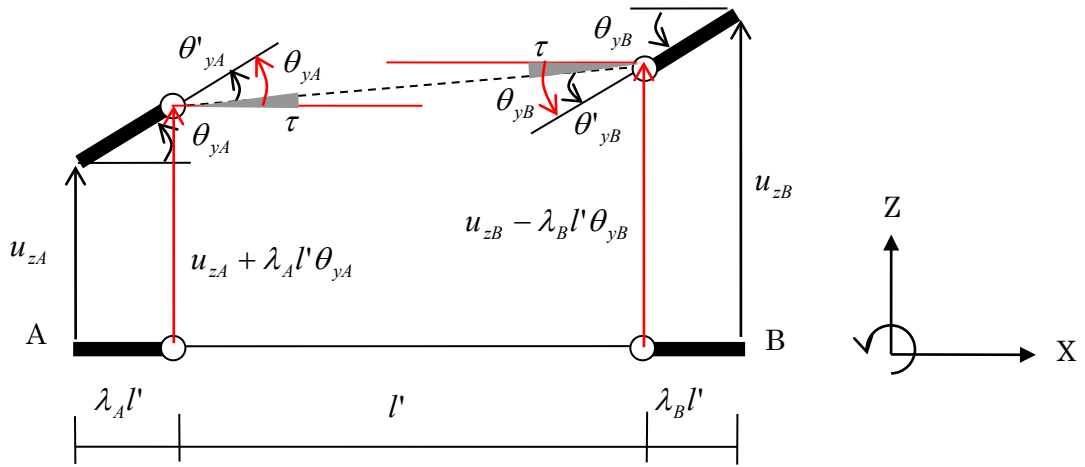


Figure 2-1-2 Including rigid parts and node movement

From node axial displacements, relative axial displacement is,

$$\delta'_x = \delta_{xB} - \delta_{xA} \quad (2-1-8)$$

Therefore

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta'_x \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = [n_B] \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} \quad (2-1-9)$$

Combining Equations (2-1-7) and (2-1-9),

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = \begin{bmatrix} \frac{1}{l'} & -\frac{1}{l'} & 1 + \lambda_A & \lambda_B & 0 & 0 \\ \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1 + \lambda_B & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = [\Lambda_B] \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} \quad (2-1-10)$$

### Out of plane deformation of beam

If we consider out-of-plane deformation of beam in case of flexible floor, as shown in Figure 2-1-4, the rotational displacement vector is,

$$\begin{Bmatrix} \theta'_{zA} \\ \theta'_{zB} \end{Bmatrix} = \begin{Bmatrix} \theta_{zA} - \tau \\ \theta_{zB} - \tau \end{Bmatrix}, \quad \tau = \frac{(u_{yA} - \lambda_A l' \theta_{zA}) - (u_{yB} + \lambda_B l' \theta_{zB})}{l'}$$

$$= \begin{Bmatrix} \theta_{zA} - \frac{1}{l'} u_{yA} + \lambda_A \theta_{zA} + \frac{1}{l'} u_{yB} + \lambda_B \theta_{zB} \\ \theta_{zB} - \frac{1}{l'} u_{yA} + \lambda_A \theta_{zA} + \frac{1}{l'} u_{yB} + \lambda_B \theta_{zB} \end{Bmatrix} = \begin{bmatrix} -\frac{1}{l'} & \frac{1}{l'} & 1 + \lambda_A & \lambda_B \\ -\frac{1}{l'} & \frac{1}{l'} & \lambda_A & 1 + \lambda_B \end{bmatrix} \begin{Bmatrix} u_{yA} \\ u_{yB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} \quad (2-1-11)$$

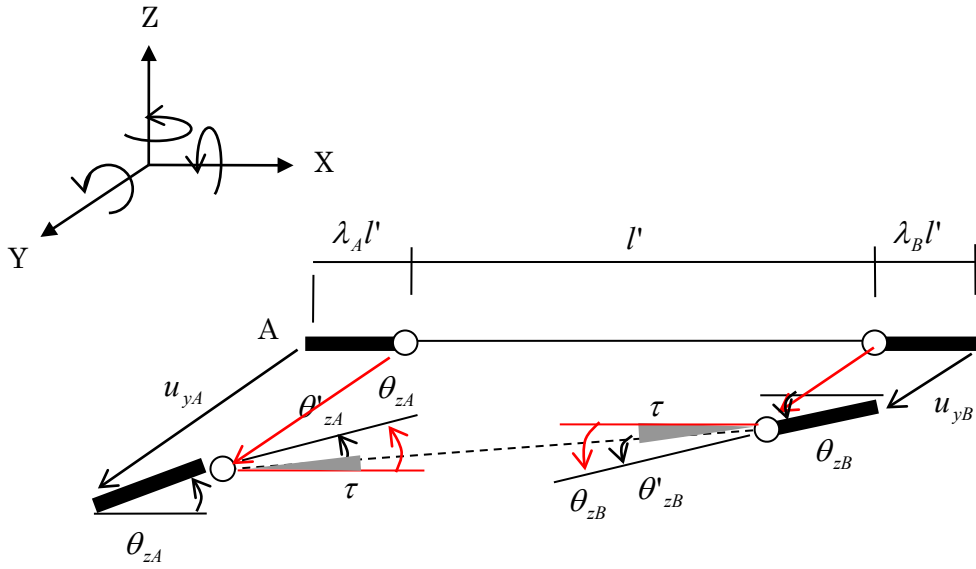


Figure 2-1-3 Beam displacement with rigid connection (X-Y plane)

We can summarize for both ends as

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{zA} \\ \theta'_{zB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = \begin{bmatrix} -\frac{1}{l'} & \frac{1}{l'} & 1 + \lambda_A & \lambda_B & & \\ -\frac{1}{l'} & \frac{1}{l'} & \lambda_A & 1 + \lambda_B & & \\ & & \frac{1}{l'} & -\frac{1}{l'} & 1 + \lambda_A & \lambda_B \\ & & \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1 + \lambda_B \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ u_{yA} \\ u_{yB} \\ \theta_{zA} \\ \theta_{zB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} \quad (2-1-12)$$

### From global node displacement to element node displacement

Transformation from global node displacements to element node displacements is,

$$\begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = [T_{ixB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-1-13)$$

The component of the transformation matrix,  $[T_{ixB}]$ , is discussed in Chapter 4 (Freedom Vector).

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta'_x \end{Bmatrix} = [n_B] [\Lambda_B] [T_{ixB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{xB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-1-14)$$

### Transformation matrix for nonlinear spring displacement

The nonlinear spring displacement vector is obtained from the element face displacement as,

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{yB} \\ s_z \end{Bmatrix} = \begin{bmatrix} f_{yA} & 0 & 0 \\ 0 & f_{yB} & 0 \\ \frac{1}{k_{sz}l'} & \frac{1}{k_{sz}l'} & 0 \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ N'_x \end{Bmatrix} \quad (2-1-15)$$

**In case of Y-direction beam**

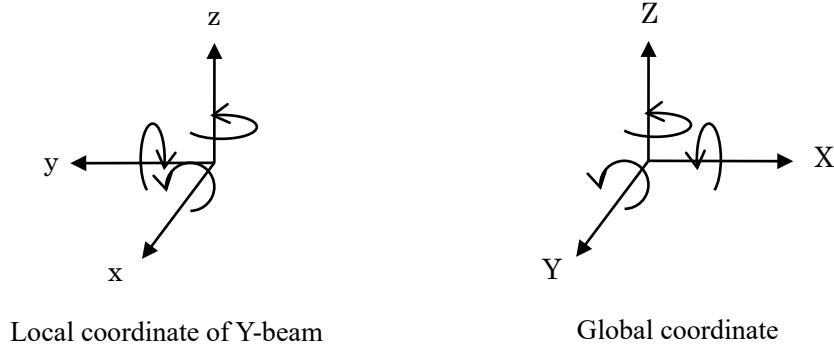


Figure 2-1-4 Relation between local coordinate and global coordinate

In case of Y-direction beam, the axial direction of the beam element coincides to the Y-axis in the global coordinate, transformation of the sign of the vector components of the element coordinate is,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{Y-Beam} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_{Global} \quad (2-1-16)$$

Therefore

$$\begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix}_{Y-Beam} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & -1 & & & \\ & & & -1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{yA} \\ \delta_{yB} \end{Bmatrix}_{Global} = [s_B] \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{yA} \\ \delta_{yB} \end{Bmatrix}_{Global} \quad (2-1-17)$$

Transformation from the global node displacement to the element node displacement is,

$$\begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{yA} \\ \delta_{yB} \end{Bmatrix} = [T_{iyB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-1-18)$$

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta'_x \end{Bmatrix} = [n_B] [\Lambda_B] [s_B] [T_{iyB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{yB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-1-19)$$

### Constitutive equation

Finally, the constitutive equation of the X-beam is,

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [T_{xB}]^T [k_B] [T_{xB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [K_{xB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-1-20)$$

For Y-beam,

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [T_{yB}]^T [k_B] [T_{yB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [K_{yB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-1-21)$$

## 2.2 Column

Element model for column is defined as a line element with nonlinear bending springs at both ends and two nonlinear shear springs in the middle of the element in x and y directions as shown in Figure 2-2-1.

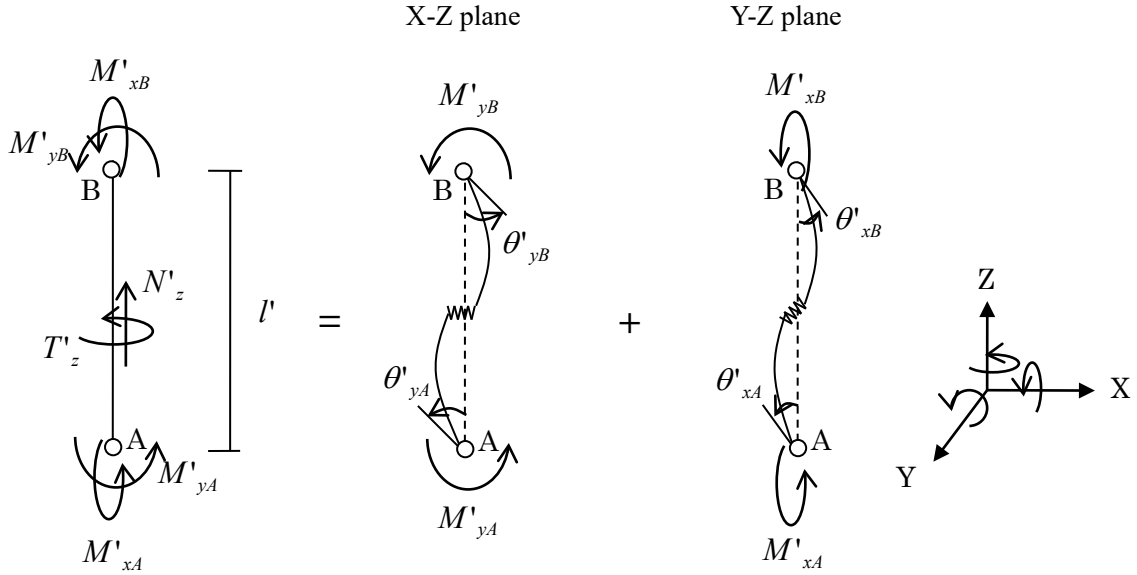


Figure 2-2-1 Element model for column

### Force-displacement relationship for elastic element

In the same way as the beam element, the relationship between the displacement vector and force vector of the elastic element is,

$$\begin{Bmatrix} \tau'_{yA} \\ \tau'_{yB} \end{Bmatrix} = \begin{bmatrix} \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} \\ -\frac{l'}{6EI_y} & \frac{l'}{3EI_y} \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \end{Bmatrix} \quad \text{in X-Z plane} \quad (2-2-1)$$

$$\begin{Bmatrix} \tau'_{xA} \\ \tau'_{xB} \end{Bmatrix} = \begin{bmatrix} \frac{l'}{3EI_x} & -\frac{l'}{6EI_x} \\ -\frac{l'}{6EI_x} & \frac{l'}{3EI_x} \end{bmatrix} \begin{Bmatrix} M'_{xA} \\ M'_{xB} \end{Bmatrix} \quad \text{in Y-Z plane} \quad (2-2-2)$$

The axial displacement is,

$$\delta''_z = \frac{l'}{EA} N'_z \quad (2-2-3)$$

The torsion angle by torque force is,

$$\theta'_z = \frac{l'}{GI_z} T'_z \quad (2-2-4)$$

where,  $G$  and  $I_z$  are the shear modulus and the pole moment of inertia of the cross-sectional area.

## Force-displacement relationship for nonlinear bending springs

**Case 1: In the case that bending springs in x and y directions are independently defined**

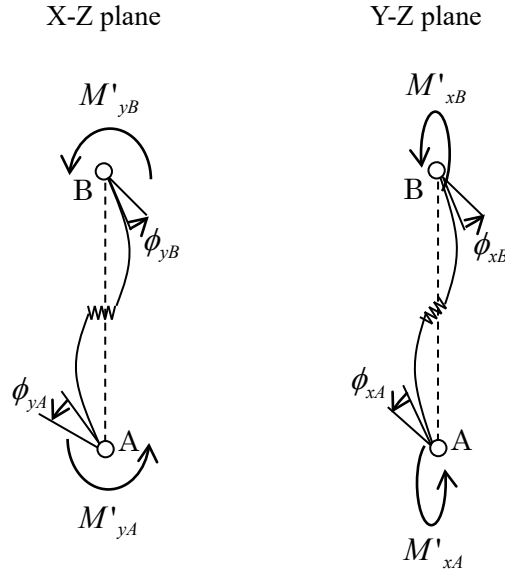


Figure 2-2-2 Element model for column

The rotational displacement vector of the nonlinear bending spring is defined independently,

$$\phi_{yA} = f_{yA} M'_{yA}, \quad \phi_{xA} = f_{xA} M'_{xA} \quad \text{at end A} \quad (2-2-5)$$

$$\phi_{yB} = f_{yB} M'_{yB}, \quad \phi_{xB} = f_{xB} M'_{xB} \quad \text{at end B} \quad (2-2-6)$$

where,  $f_{yA}$ ,  $f_{xA}$ ,  $f_{yB}$ , and  $f_{xB}$  are the flexural stiffness of nonlinear bending springs at both ends of the element.

It can be expressed as

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \end{Bmatrix} = [f_{pA}] \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \end{Bmatrix}, \quad [f_{pA}] = \begin{bmatrix} f_{yA} & & \\ & f_{xA} & \\ & & 0 \end{bmatrix} \quad \text{at end A} \quad (2-2-7)$$

$$\begin{Bmatrix} \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = [f_{pB}] \begin{Bmatrix} M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix}, \quad [f_{pB}] = \begin{bmatrix} f_{yB} & & \\ & f_{xB} & \\ & & 0 \end{bmatrix} \quad \text{at end B} \quad (2-2-8)$$

**Case 2: In the case that nonlinear interaction between moment and axial components is considered**

Nonlinear interaction  $M_x - M_y - N_z$  is considered in the nonlinear bending springs,

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \end{Bmatrix} = [f_{pA}] \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \end{Bmatrix} \quad \text{at end A} \quad (2-2-9)$$

$$\begin{Bmatrix} \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = [f_{pB}] \begin{Bmatrix} M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} \quad \text{at end B} \quad (2-2-10)$$

where,  $[f_{pA}]$  and  $[f_{pB}]$  are the flexural stiffness matrices of the nonlinear bending springs.

Therefore, the force-displacement relationship of nonlinear bending springs is,

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \\ \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \\ M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} \quad (2-2-11)$$

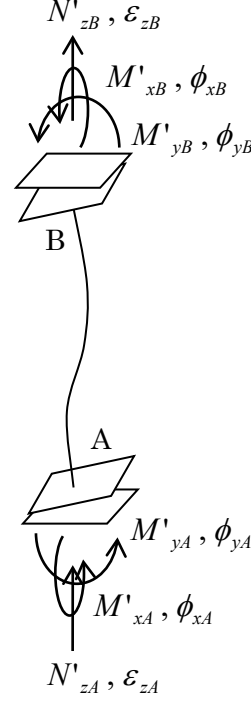


Figure 2-2-3 Nonlinear bending springs

Rearrange the order of the components of the displacement vector and change the node axial displacements into the relative axial displacement,

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{yB} \\ \phi_{xA} \\ \phi_{xB} \\ \varepsilon_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \\ \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = [n_p] \begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \\ \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} \quad (2-2-12)$$

The force-displacement relationship is then expressed as,

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{yB} \\ \phi_{xA} \\ \phi_{xB} \\ \varepsilon_z \end{Bmatrix} = [n_p] \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} [n_p]^T \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ M'_{xA} \\ M'_{xB} \\ N'_z \end{Bmatrix} = [f_p] \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ M'_{xA} \\ M'_{xB} \\ N'_z \end{Bmatrix} \quad (2-2-13)$$

## Force-displacement relationship for nonlinear shear springs

**Case 1: In the case that shear springs in x and y directions are independently defined**

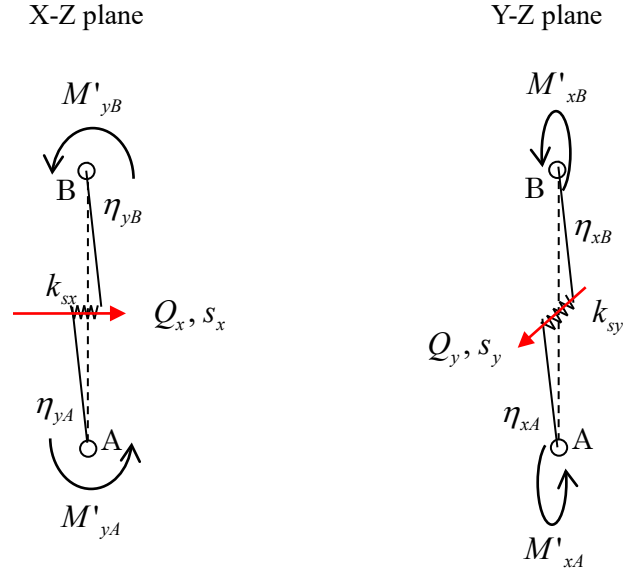


Figure 2-2-4 Element model for column

The force-deformation relationship of shear spring is

$$\begin{Bmatrix} Q'_x \\ Q'_y \end{Bmatrix} = \begin{bmatrix} k_{sx} & 0 \\ 0 & k_{sy} \end{bmatrix} \begin{Bmatrix} s_x \\ s_y \end{Bmatrix}, \quad \begin{Bmatrix} s_x \\ s_y \end{Bmatrix} = \begin{bmatrix} 1/k_{sx} & 0 \\ 0 & 1/k_{sy} \end{bmatrix} \begin{Bmatrix} Q'_x \\ Q'_y \end{Bmatrix} \quad (2-2-14)$$

From the relationship between shear force and moment,

$$\begin{Bmatrix} Q'_x \\ Q'_y \end{Bmatrix} = \begin{bmatrix} 1/l' & 1/l' & 0 & 0 \\ 0 & 0 & 1/l' & 1/l' \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ M'_{xA} \\ M'_{xB} \end{Bmatrix} \quad (2-2-15)$$

The end rotational displacement due to shear deformation is obtained as,

$$\begin{aligned} \begin{Bmatrix} \eta_{yA} \\ \eta_{yB} \\ \eta_{xA} \\ \eta_{xB} \end{Bmatrix} &= \begin{bmatrix} 1/l' & 0 \\ 1/l' & 0 \\ 0 & 1/l' \\ 0 & 1/l' \end{bmatrix} \begin{Bmatrix} s_x \\ s_y \end{Bmatrix} = \begin{bmatrix} 1/l' & 0 \\ 1/l' & 0 \\ 0 & 1/l' \\ 0 & 1/l' \end{bmatrix} \begin{bmatrix} 1/k_{sx} & 0 \\ 0 & 1/k_{sy} \end{bmatrix} \begin{Bmatrix} Q'_x \\ Q'_y \end{Bmatrix} = \begin{bmatrix} 1/(k_{sx}l') & 0 \\ 1/(k_{sx}l') & 0 \\ 0 & 1/(k_{sy}l') \\ 0 & 1/(k_{sy}l') \end{bmatrix} \begin{Bmatrix} Q'_x \\ Q'_y \end{Bmatrix} \\ &= \begin{bmatrix} 1/(k_{sx}l') & 0 \\ 1/(k_{sx}l') & 0 \\ 0 & 1/(k_{sy}l') \\ 0 & 1/(k_{sy}l') \end{bmatrix} \begin{bmatrix} 1/l' & 1/l' & 0 & 0 \\ 0 & 0 & 1/l' & 1/l' \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ M'_{xA} \\ M'_{xB} \end{Bmatrix} = [f_{s1}] \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ M'_{xA} \\ M'_{xB} \end{Bmatrix} \quad (2-2-16) \end{aligned}$$

where  $[f_{s1}] = \begin{bmatrix} \frac{1}{k_{sx}l'^2} & \frac{1}{k_{sx}l'^2} & 0 & 0 \\ \frac{1}{k_{sx}l'^2} & \frac{1}{k_{sx}l'^2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_{sy}l'^2} & \frac{1}{k_{sy}l'^2} \\ 0 & 0 & \frac{1}{k_{sy}l'^2} & \frac{1}{k_{sy}l'^2} \end{bmatrix}$  (2-2-17)

**Case 2: In the case that nonlinear interaction between shear and axial components is considered**

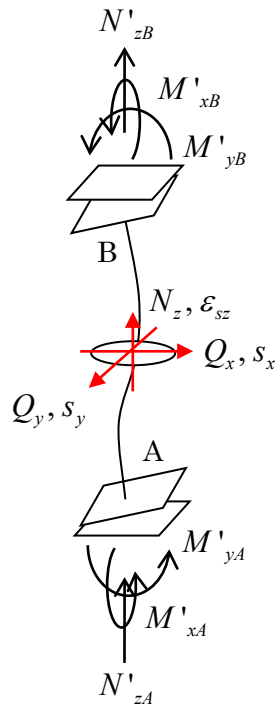


Figure 2-2-5 Nonlinear shear springs

The force-deformation relationship of shear spring is

$$\begin{Bmatrix} Q'_x \\ Q'_y \\ N'_z \end{Bmatrix} = [k_{sp}] \begin{Bmatrix} s_x \\ s_y \\ \epsilon_{sz} \end{Bmatrix} = [k_{sp}] \begin{bmatrix} l' & & \\ & l' & \\ & & 1 \end{bmatrix} \begin{Bmatrix} \eta_y \\ \eta_x \\ \epsilon_{sz} \end{Bmatrix} = [k'_{sp}] \begin{Bmatrix} \eta_y \\ \eta_x \\ \epsilon_{sz} \end{Bmatrix} \quad (2-2-18)$$

$$\begin{Bmatrix} \eta_y \\ \eta_x \\ \epsilon_{sz} \end{Bmatrix} = [f_{sp}] \begin{Bmatrix} Q'_x \\ Q'_y \\ N'_z \end{Bmatrix}, \quad [f_{sp}] = [k'_{sp}]^{-1} \quad (2-2-19)$$

From the relationship between shear force and moment,

$$\begin{Bmatrix} Q'_x \\ Q'_y \\ N'_z \end{Bmatrix} = \begin{bmatrix} 1/l' & 1/l' & 0 & 0 & 0 \\ 0 & 0 & 1/l' & 1/l' & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N_z \end{Bmatrix} = [L_s] \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N_z \end{Bmatrix} \quad (2-2-20)$$

The end rotational displacement due to shear deformation is obtained as,

$$\begin{Bmatrix} \eta_{yA} \\ \eta_{yB} \\ \eta_{xA} \\ \eta_{xB} \\ \varepsilon_{sz} \end{Bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{Bmatrix} \eta_y \\ \eta_x \\ \varepsilon_{sz} \end{Bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} [f_{sp}] [L_s] \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N_z \end{Bmatrix} = [f_{s2}] \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N_z \end{Bmatrix} \quad (2-2-21)$$

where

$$\begin{bmatrix} 1 & & & & \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} [L_s] = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} 1/l' & 1/l' & 0 & 0 & 0 \\ 0 & 0 & 1/l' & 1/l' & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (2-2-22)$$

$$= \begin{bmatrix} f_{11}/l' & f_{11}/l' & f_{12}/l' & f_{12}/l' & f_{13} \\ f_{11}/l' & f_{11}/l' & f_{12}/l' & f_{12}/l' & f_{13} \\ f_{21}/l' & f_{21}/l' & f_{22}/l' & f_{22}/l' & f_{23} \\ f_{21}/l' & f_{21}/l' & f_{22}/l' & f_{22}/l' & f_{23} \\ f_{31}/l' & f_{31}/l' & f_{32}/l' & f_{32}/l' & f_{33} \end{bmatrix}$$

Both cases can be written in

$$\begin{Bmatrix} \eta_{yA} \\ \eta_{yB} \\ \eta_{xA} \\ \eta_{xB} \\ e_{sz} \end{Bmatrix} = [f_s] \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N_z \end{Bmatrix}, \quad [f_s] = \begin{bmatrix} [f_{s1}] & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{case 1}), \quad [f_s] = [f_{s2}] \quad (\text{case 2}) \quad (2-2-23)$$

The displacement vector of the column element is obtained as the sum of the displacement vectors of elastic element, nonlinear shear springs and nonlinear bending springs,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta'_z \\ \theta'_z \end{Bmatrix} = \underbrace{\begin{Bmatrix} \tau'_{yA} \\ \tau'_{yB} \\ \tau'_{xA} \\ \tau'_{xB} \\ \delta''_z \\ \theta'_z \end{Bmatrix}}_{\text{elastic element}} + \underbrace{\begin{Bmatrix} \varphi_{yA} \\ \varphi_{yB} \\ \varphi_{xA} \\ \varphi_{xB} \\ \varepsilon_z \\ 0 \end{Bmatrix}}_{\text{bending spring}} + \underbrace{\begin{Bmatrix} \eta_{yA} \\ \eta_{yB} \\ \eta_{xA} \\ \eta_{xB} \\ \varepsilon_{sz} \\ 0 \end{Bmatrix}}_{\text{shear spring}} = [f_C] \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ M'_{xA} \\ M'_{xB} \\ N'_z \\ T'_z \end{Bmatrix} \quad (2-2-24)$$

The flexural matrix  $[f_C]$  is;

$$\begin{aligned}
[f_C] = & \begin{bmatrix} \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & & & & \\ & \frac{l'}{3EI_y} & & & & \\ & & \frac{l'}{3EI_x} & -\frac{l'}{6EI_x} & & \\ & & & \frac{l'}{3EI_x} & & \\ & & & & \frac{l'}{EA} & \\ \text{sym.} & & & & & \frac{l'}{GI_z} \end{bmatrix}_{\text{elastic element}} + \\
& \begin{bmatrix} f_{p11} & f_{p12} & f_{p13} & f_{p14} & f_{p15} & 0 \\ & f_{p22} & f_{p23} & f_{p24} & f_{p25} & 0 \\ & & f_{p33} & f_{p34} & f_{p35} & 0 \\ & & & f_{p44} & f_{p45} & 0 \\ & & & & f_{p55} & 0 \\ \text{sym.} & & & & & 0 \end{bmatrix}_{\text{bending spring}} + \\
& \begin{bmatrix} [f_s] & 0 \\ 0 & 0 \end{bmatrix}_{\text{shear spring}} \quad (2-2-25)
\end{aligned}$$

By taking the inverse matrix of  $[f_C]$ , the constitutive equation of the column element is obtained as,

$$\begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ M'_{xA} \\ M'_{xB} \\ N'_z \\ T'_z \end{Bmatrix} = [f_C]^{-1} \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta'_z \\ \theta'_z \end{Bmatrix} = [k_C] \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta'_z \\ \theta'_z \end{Bmatrix} \quad (2-2-26)$$

### Including rigid parts and node movement

Change relative axial displacement and torsion displacement into node displacement,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta'_z \\ \theta'_z \end{Bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ 0 & & & & -1 & 1 \\ & & & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} = [n_c] \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} \quad (2-2-27)$$

Including rigid parts and node movement,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} = \begin{bmatrix} -\frac{1}{l'} & \frac{1}{l'} & 1+\lambda_A & \lambda_B & & & & \\ -\frac{1}{l'} & \frac{1}{l'} & \lambda_A & 1+\lambda_B & & & & \\ & & & & \frac{1}{l'} & -\frac{1}{l'} & 1+\lambda_A & \lambda_B \\ & & & & \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1+\lambda_B \\ & & & & & & & 1 \\ & & & & & & & 1 \\ & & & & & & & 1 \\ 0 & & & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \\ u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} = [\Lambda_c] \begin{Bmatrix} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \\ u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} \quad (2-2-28)$$

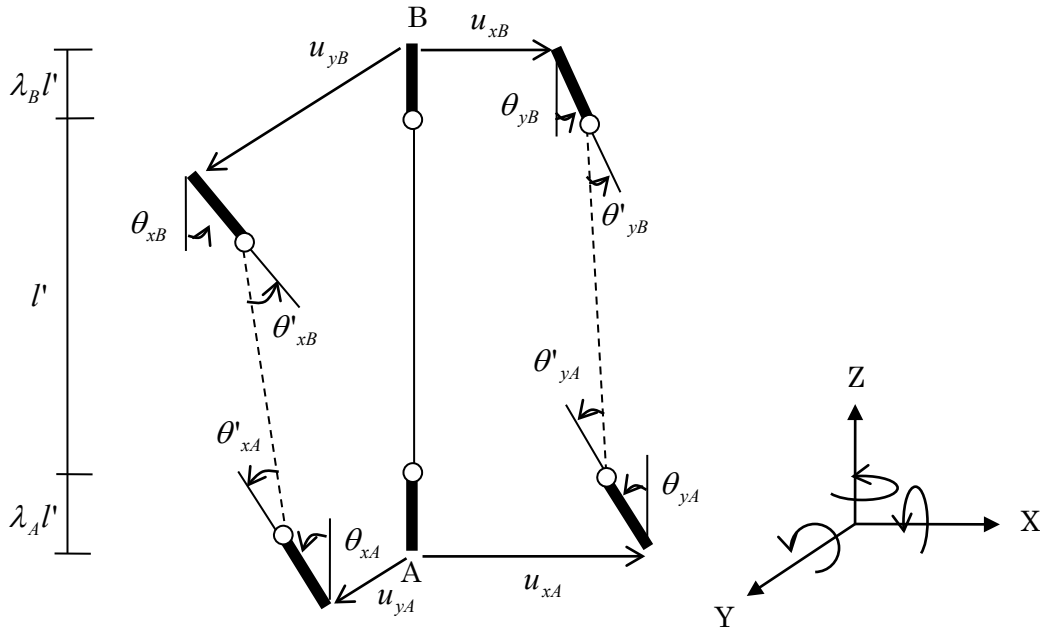


Figure 2-2-6 Including rigid parts and node movement

### From global node displacement to element node displacement

Transformation from global node displacement to element node displacement is;

$$\begin{Bmatrix} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \\ u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} = [T_{iC}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-2-29)$$

The component of the transformation matrix,  $[T_{iC}]$ , is discussed in Chapter 4 (Freedom Vector).

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta'_z \\ \theta'_z \end{Bmatrix} = [n_C] [\Lambda_C] [T_{iC}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_C] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-2-30)$$

### Constitutive equation

Finally, the constitutive equation of the column is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_C] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-2-31)$$

where,

$$[K_C] = [T_C]^T [k_C] [T_C] \quad (2-2-32)$$

### Transformation matrix for nonlinear spring displacement

The nonlinear spring displacement vector is obtained from

$$\begin{Bmatrix} \varphi_{yA} \\ \varphi_{xA} \\ \varepsilon_{zA} \\ \varphi_{yB} \\ \varphi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \\ M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{bmatrix} [n_p]^T & 0 \end{bmatrix} \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N'_z \\ T'_z \end{Bmatrix} = [f_{mC}] \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N'_z \\ T'_z \end{Bmatrix} \quad (2-2-33)$$

and

$$\begin{Bmatrix} \eta_y \\ \eta_x \\ \varepsilon_{sz} \end{Bmatrix} = [f_{sp}] \begin{Bmatrix} Q'_x \\ Q'_y \\ N'_z \end{Bmatrix} = [f_{sp}] [L_s] \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N'_z \end{Bmatrix} = [f_{sp}] [L_s] \begin{bmatrix} [I] & 0 \end{bmatrix} \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N'_z \\ T'_z \end{Bmatrix} = [f_{sC}] \begin{Bmatrix} M_{yA} \\ M_{yB} \\ M_{xA} \\ M_{xB} \\ N'_z \\ T'_z \end{Bmatrix} \quad (2-2-34)$$

## 2.3 Wall

Element model for wall is defined as a line element with nonlinear bending springs at both ends and three nonlinear shear springs; one is in the middle of the wall panel and others are in the side columns as shown in Figure 2-3-1.

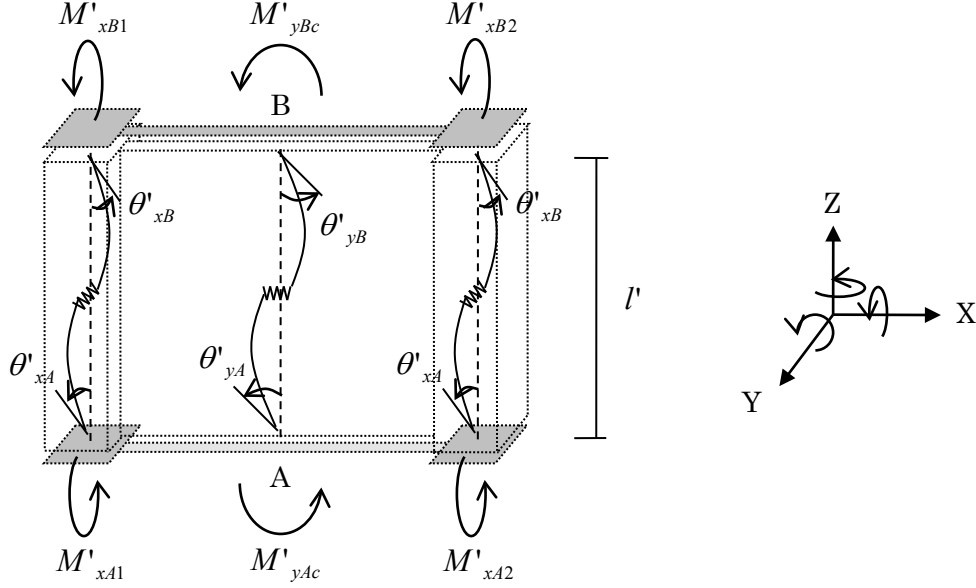


Figure 2-3-1 Element model for wall

### Force-displacement relationship for elastic element

In the same way as the beam element, the relationship between the displacement vector and force vector of the elastic element is,

$$\begin{Bmatrix} \tau'_{yAc} \\ \tau'_{yBc} \end{Bmatrix} = \begin{bmatrix} \frac{l'}{3EI_c} & -\frac{l'}{6EI_c} \\ -\frac{l'}{6EI_c} & \frac{l'}{3EI_c} \end{bmatrix} \begin{Bmatrix} M'_{yAc} \\ M'_{yBc} \end{Bmatrix} \quad \text{in wall panel} \quad (2-3-1)$$

$$\begin{Bmatrix} \tau'_{xA1} \\ \tau'_{xB1} \end{Bmatrix} = \begin{bmatrix} \frac{l'}{3EI_1} & -\frac{l'}{6EI_1} \\ -\frac{l'}{6EI_1} & \frac{l'}{3EI_1} \end{bmatrix} \begin{Bmatrix} M'_{xA1} \\ M'_{xB1} \end{Bmatrix} \quad \text{in side column 1} \quad (2-3-2)$$

$$\begin{Bmatrix} \tau'_{xA2} \\ \tau'_{xB2} \end{Bmatrix} = \begin{bmatrix} \frac{l'}{3EI_2} & -\frac{l'}{6EI_2} \\ -\frac{l'}{6EI_2} & \frac{l'}{3EI_2} \end{bmatrix} \begin{Bmatrix} M'_{xA2} \\ M'_{xB2} \end{Bmatrix} \quad \text{in side column 2} \quad (2-3-3)$$

The axial displacement is,

$$\delta''_{zc} = \frac{l'}{EA} N'_{zc} \quad (2-3-4)$$

### Force-displacement relationship for nonlinear bending springs

Nonlinear interaction  $M_x - M_y - N_z$  is considered in the nonlinear bending springs,

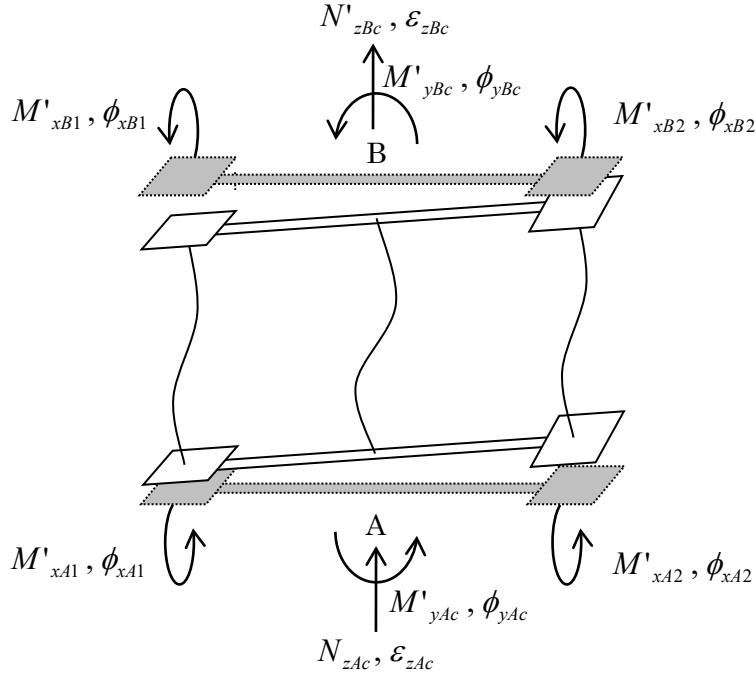


Figure 2-3-2 Nonlinear bending springs

$$\begin{Bmatrix} \phi_{yAc} \\ \phi_{xA1} \\ \phi_{xA2} \\ \epsilon_{zAc} \end{Bmatrix} = [f_{pA}] \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \end{Bmatrix} \quad \text{at end A} \quad (2-3-5)$$

$$\begin{Bmatrix} \phi_{yBc} \\ \phi_{xB1} \\ \phi_{xB2} \\ \epsilon_{zBc} \end{Bmatrix} = [f_{pB}] \begin{Bmatrix} M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix} \quad \text{at end B} \quad (2-3-6)$$

where,  $[f_{pA}]$  and  $[f_{pB}]$  are the flexural stiffness matrices of the nonlinear bending springs. Therefore, the force-displacement relationship of nonlinear bending springs is,

$$\begin{Bmatrix} \phi_{yAc} \\ \phi_{xA1} \\ \phi_{xA2} \\ \varepsilon_{zAc} \\ \phi_{yBc} \\ \phi_{xB1} \\ \phi_{xB2} \\ \varepsilon_{zBc} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix} \quad (2-3-7)$$

Rearrange the order of the components of the displacement vector and change the node axial displacements into the relative axial displacement,

$$\begin{Bmatrix} \phi_{yAc} \\ \phi_{yBc} \\ \phi_{xA1} \\ \phi_{xB1} \\ \phi_{xA2} \\ \phi_{xB2} \\ \varepsilon_{zc} \end{Bmatrix} = \begin{bmatrix} 1 & & & & & & \\ & & 1 & & & & \\ & 1 & & & & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & & & & 1 & & \\ & & & -1 & & 1 & \end{bmatrix} \begin{Bmatrix} \phi_{yAc} \\ \phi_{xA1} \\ \phi_{xA2} \\ \varepsilon_{zAc} \\ \phi_{yBc} \\ \phi_{xB1} \\ \phi_{xB2} \\ \varepsilon_{zBc} \end{Bmatrix} = [n_p] \begin{Bmatrix} \phi_{yAc} \\ \phi_{xA1} \\ \phi_{xA2} \\ \varepsilon_{zAc} \\ \phi_{yBc} \\ \phi_{xB1} \\ \phi_{xB2} \\ \varepsilon_{zBc} \end{Bmatrix} \quad (2-3-8)$$

The force-displacement relationship in Equation (2-3-7) is then expressed as,

$$\begin{Bmatrix} \phi_{yAc} \\ \phi_{yBc} \\ \phi_{xA1} \\ \phi_{xB1} \\ \phi_{xA2} \\ \phi_{xB2} \\ \varepsilon_{zc} \end{Bmatrix} = [n_p] \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} [n_p]^T \begin{Bmatrix} M'_{yAc} \\ M'_{yBc} \\ M'_{xA1} \\ M'_{xB1} \\ M'_{xA2} \\ M'_{xB2} \\ N'_{zc} \end{Bmatrix} = [f_p] \begin{Bmatrix} M'_{yAc} \\ M'_{yBc} \\ M'_{xA1} \\ M'_{xB1} \\ M'_{xA2} \\ M'_{xB2} \\ N'_{zc} \end{Bmatrix} \quad (2-3-9)$$

### Force-displacement relationship for nonlinear shear springs

The force-deformation relationship of shear spring in the center is

$$Q'_{xc} = k_{sc} s_{xc}, \quad s_{xc} = (1/k_{sc}) Q'_{xc}$$

$$Q'_{xc} = [1/l' \quad 1/l'] \begin{Bmatrix} M'_{yAc} \\ M'_{yBc} \end{Bmatrix}$$

The end rotational displacement due to shear deformation is obtained as,

$$\begin{Bmatrix} \eta_{yAc} \\ \eta_{yBc} \end{Bmatrix} = \begin{bmatrix} 1/l' \\ 1/l' \end{bmatrix} s_{xc} = \begin{bmatrix} 1/l' \\ 1/l' \end{bmatrix} (1/k_{sc}) Q'_{xc} = \begin{bmatrix} \frac{1}{k_{sc} l'^2} & \frac{1}{k_{sc} l'^2} \\ \frac{1}{k_{sc} l'^2} & \frac{1}{k_{sc} l'^2} \end{bmatrix} \begin{Bmatrix} M'_{xAc} \\ M'_{xBc} \end{Bmatrix} \quad \text{in wall panel} \quad (2-3-10)$$

$$\begin{Bmatrix} \eta_{xA1} \\ \eta_{xB1} \end{Bmatrix} = \begin{bmatrix} \frac{1}{k_{s1} l'^2} & \frac{1}{k_{s1} l'^2} \\ \frac{1}{k_{s1} l'^2} & \frac{1}{k_{s1} l'^2} \end{bmatrix} \begin{Bmatrix} M'_{xA1} \\ M'_{xB1} \end{Bmatrix} \quad \text{in side column 1} \quad (2-3-11)$$

$$\begin{Bmatrix} \eta_{xA2} \\ \eta_{xB2} \end{Bmatrix} = \begin{bmatrix} \frac{1}{k_{s2} l'^2} & \frac{1}{k_{s2} l'^2} \\ \frac{1}{k_{s2} l'^2} & \frac{1}{k_{s2} l'^2} \end{bmatrix} \begin{Bmatrix} M'_{xA2} \\ M'_{xB2} \end{Bmatrix} \quad \text{in side column 2} \quad (2-3-12)$$

where,  $k_{sc}$ ,  $k_{s1}$  and  $k_{s2}$  are the shear stiffness of the nonlinear shear springs.

The displacement vector of the column element is obtained as the sum of the displacement vectors of elastic element, nonlinear shear springs and nonlinear bending springs,

$$\begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zc} \end{Bmatrix} = \underbrace{\begin{Bmatrix} \tau'_{yAc} \\ \tau'_{yBc} \\ \tau'_{xA1} \\ \tau'_{xB1} \\ \tau'_{xA2} \\ \tau'_{xB2} \\ \delta''_{zc} \end{Bmatrix}}_{\text{elastic element}} + \underbrace{\begin{Bmatrix} \phi_{yAc} \\ \phi_{yBc} \\ \phi_{xA1} \\ \phi_{xB1} \\ \phi_{xA2} \\ \phi_{xB2} \\ \varepsilon_{zc} \end{Bmatrix}}_{\text{bending spring}} + \underbrace{\begin{Bmatrix} \eta_{yAc} \\ \eta_{yBc} \\ \eta_{xA1} \\ \eta_{xB1} \\ \eta_{xA2} \\ \eta_{xB2} \\ 0 \end{Bmatrix}}_{\text{shear spring}} = [f_w] \begin{Bmatrix} M'_{yAc} \\ M'_{yBc} \\ M'_{xA1} \\ M'_{xB1} \\ M'_{xA2} \\ M'_{xB2} \\ N'_{zc} \end{Bmatrix} \quad (2-3-13)$$

The flexural matrix  $[f_w]$  is;

$$\begin{aligned}
[f_W] = & \begin{bmatrix} \frac{l'}{3EI_c} & -\frac{l'}{6EI_c} & & & & & \\ & \frac{l'}{3EI_c} & & & & & \\ & & \frac{l'}{3EI_1} & -\frac{l'}{6EI_1} & & & \\ & & & \frac{l'}{3EI_1} & & & \\ & & & & \frac{l'}{3EI_2} & -\frac{l'}{6EI_2} & \\ & & & & & \frac{l'}{3EI_2} & \\ & \text{sym.} & & & & & \frac{l'}{EA_c} \end{bmatrix}_{\text{elastic element}} + \\
& \begin{bmatrix} f_{p11} & \cdots & f_{p17} \\ \vdots & & \vdots \\ f_{p71} & \cdots & f_{p77} \end{bmatrix}_{\text{bending spring}} + \\
& \begin{bmatrix} \frac{1}{k_{sc}l^2} & \frac{1}{k_{sc}l^2} & & & & & \\ & \frac{1}{k_{sc}l^2} & & & & & \\ & & \frac{1}{k_{s1}l^2} & \frac{1}{k_{s1}l^2} & & & \\ & & & \frac{1}{k_{s1}l^2} & & & \\ & & & & \frac{1}{k_{s2}l^2} & \frac{1}{k_{s2}l^2} & \\ & & & & & \frac{1}{k_{s2}l^2} & \\ & \text{sym.} & & & & & 0 \end{bmatrix}_{\text{shear spring}} \quad (2-3-14)
\end{aligned}$$

By taking the inverse matrix of  $[f_W]$ , the constitutive equation of the column element is obtained as,

$$\begin{Bmatrix} M'_{yAc} \\ M'_{yBc} \\ M'_{xA1} \\ M'_{xB1} \\ M'_{xA2} \\ M'_{xB2} \\ N'_{zc} \end{Bmatrix} = [f_W]^{-1} \begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zc} \end{Bmatrix} = [k_W] \begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zc} \end{Bmatrix} \quad (2-3-15)$$

### Including rigid parts and node movement

Change relative axial displacement and torsion displacement into node displacement,

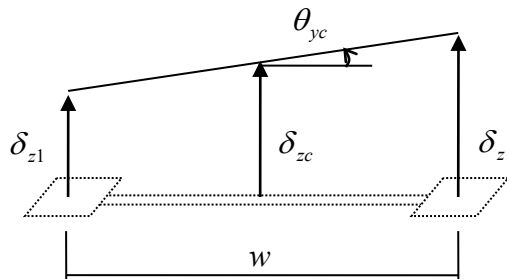
$$\begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zAc} \\ \delta'_{zBc} \end{Bmatrix} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zAc} \\ \delta'_{zBc} \end{Bmatrix} = [n_w] \begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zAc} \\ \delta'_{zBc} \end{Bmatrix} \quad (2-3-16)$$

Including rigid parts and node movement,

$$\begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zAc} \\ \delta'_{zBc} \end{Bmatrix} = \begin{bmatrix} -\frac{1}{l'} & \frac{1}{l'} & 1+\lambda_A & \lambda_B & & & & \\ -\frac{1}{l'} & \frac{1}{l'} & \lambda_A & 1+\lambda_B & & & & \\ & & & & \frac{1}{l'} & -\frac{1}{l'} & 1+\lambda_A & \lambda_B \\ & & & & \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1+\lambda_B \\ & & & & & & \frac{1}{l'} & -\frac{1}{l'} & 1+\lambda_A & \lambda_B \\ & & & & & & \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1+\lambda_B \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{xAc} \\ u_{xBc} \\ \theta_{yAc} \\ \theta_{yBc} \\ u_{yA1} \\ u_{yB1} \\ \theta_{xA1} \\ \theta_{xB1} \\ u_{yA2} \\ u_{yB2} \\ \theta_{xA2} \\ \theta_{xB2} \\ \delta_{zAc} \\ \delta_{zBc} \end{Bmatrix} = [\Lambda_w] \begin{Bmatrix} u_{xAc} \\ u_{xBc} \\ \theta_{yAc} \\ \theta_{yBc} \\ u_{yA1} \\ u_{yB1} \\ \theta_{xA1} \\ \theta_{xB1} \\ u_{yA2} \\ u_{yB2} \\ \theta_{xA2} \\ \theta_{xB2} \\ \delta_{zAc} \\ \delta_{zBc} \end{Bmatrix} \quad (2-3-17)$$

### From global node displacement to element node displacement

Transformation from the center displacements to the node displacements is,



$$\theta_{yc} = \frac{\delta_{z2} - \delta_{z1}}{w}$$

$$\delta_{zc} = \frac{\delta_{z1} + \delta_{z2}}{2}$$

Figure 2-3-3 Relationship between center and node displacements

$$\begin{Bmatrix} u_{xAc} \\ u_{xBc} \\ \theta_{yAc} \\ \theta_{yBc} \\ u_{yA1} \\ u_{yB1} \\ \theta_{xA1} \\ \theta_{xB1} \\ u_{yA2} \\ u_{yB2} \\ \theta_{xA2} \\ \theta_{xB2} \\ \delta_{zAc} \\ \delta_{zBc} \end{Bmatrix} = \begin{bmatrix} 1 & & & & & & & & & & & & & \\ & -\frac{1}{w} & \frac{1}{w} & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & \\ & & & & -\frac{1}{w} & \frac{1}{w} & & & & & & & & \\ & & & & & & 1 & & & & & & & \\ & & & & & & & 1 & & & & & & \\ & & & & & & & & 1 & & & & & \\ & & & & & & & & & 1 & & & & \\ & & & & & & & & & & 1 & & & \\ & & & & & & & & & & & 1 & & \\ & & & & & & & & & & & & 1 & \\ & 0.5 & 0.5 & & & & & & & & & & & \\ & & & 0.5 & 0.5 & & & & & & & & & \end{bmatrix} \begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ \delta_{zB2} \\ u_{yA1} \\ u_{yB1} \\ \theta_{xA1} \\ \theta_{xB1} \\ u_{yA2} \\ u_{yB2} \\ \theta_{xA2} \\ \theta_{xB2} \end{Bmatrix} = [D_W] \begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ \delta_{zB2} \\ u_{yA1} \\ u_{yB1} \\ \theta_{xA1} \\ \theta_{xB1} \\ u_{yA2} \\ u_{yB2} \\ \theta_{xA2} \\ \theta_{xB2} \end{Bmatrix} \quad (2-3-18)$$

Transformation from the global node displacements to the element node displacements is;

$$\begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ \delta_{zB2} \\ u_{yA1} \\ u_{yB1} \\ \theta_{xA1} \\ \theta_{xB1} \\ u_{yA2} \\ u_{yB2} \\ \theta_{xA2} \\ \theta_{xB2} \end{Bmatrix} = [T_{ixW}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-19)$$

The component of the transformation matrix,  $[T_{ixW}]$ , is discussed in Chapter 4 (Freedom Vector).

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zc} \end{Bmatrix} = [n_w] [\Lambda_w] [D_w] [T_{ixw}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{xw}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-20)$$

**In case of Y-direction wall**

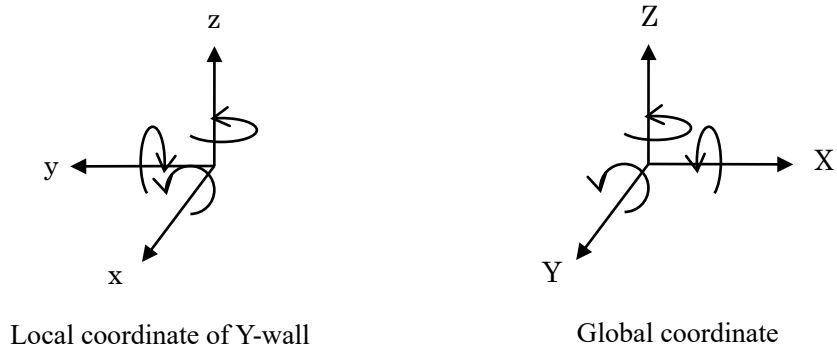


Figure 2-3-4 Relation between local coordinate and global coordinate

In case of Y-direction wall, the wall panel direction coincides to the Y-axis in the global coordinate, transformation of the sign of the vector components of the element coordinate is,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{Y-Wall} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_{Global} \quad (2-3-21)$$

Therefore

$$\begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ \delta_{zB2} \\ u_{yA1} \\ u_{yB1} \\ \theta_{xA1} \\ \theta_{xB1} \\ u_{yA2} \\ u_{yB2} \\ \theta_{xA2} \\ \theta_{xB2} \end{Bmatrix}_{Y-Wall} = \begin{bmatrix} 1 & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & \\ & & 1 & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & \\ & & & & 1 & & & & & & & & & \\ & & & & & 1 & & & & & & & & \\ & & & & & & 1 & & & & & & & \\ & & & & & & & -1 & & & & & & \\ & & & & & & & & -1 & & & & & \\ & & & & & & & & & 1 & & & & \\ & & & & & & & & & & 1 & & & \\ & & & & & & & & & & & -1 & & \\ & & & & & & & & & & & & -1 & \\ & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ \delta_{zB2} \\ u_{xA1} \\ u_{xB1} \\ \theta_{yA1} \\ \theta_{yB1} \\ u_{xA2} \\ u_{xB2} \\ \theta_{yA2} \\ \theta_{yB2} \end{Bmatrix}_{Global} = [\varepsilon_W] \begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ \delta_{zB2} \\ u_{xA1} \\ u_{xB1} \\ \theta_{yA1} \\ \theta_{yB1} \\ u_{xA2} \\ u_{xB2} \\ \theta_{yA2} \\ \theta_{yB2} \end{Bmatrix}_{Global} \quad (2-3-22)$$

Transformation from the global node displacement to the element node displacement is;

$$\begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ \delta_{zB2} \\ u_{xA1} \\ u_{xB1} \\ \theta_{yA1} \\ \theta_{yB1} \\ u_{xA2} \\ u_{xB2} \\ \theta_{yA2} \\ \theta_{yB2} \end{Bmatrix} = [T_{iyW}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-23)$$

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \theta'_{yAc} \\ \theta'_{yBc} \\ \theta'_{xA1} \\ \theta'_{xB1} \\ \theta'_{xA2} \\ \theta'_{xB2} \\ \delta'_{zc} \end{Bmatrix} = [n_W][\Lambda_W][D_W][\varepsilon_W][T_{ixW}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{yW}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-24)$$

### Constitutive equation

Finally, the constitutive equation of the wall is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{xW}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-25)$$

where,

$$[K_{xW}] = [T_{xW}]^T [k_W][T_{xW}] \quad (2-3-26)$$

For Y-wall,

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{yW}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-27)$$

where,

$$[K_{yW}] = [T_{yW}]^T [k_W][T_{yW}] \quad (2-3-28)$$

### Transformation matrix for nonlinear spring displacement

The nonlinear spring displacement vector is obtained from

$$\begin{Bmatrix} \varphi_{yAc} \\ \varphi_{xA1} \\ \varphi_{xA2} \\ \varepsilon_{zAc} \\ \varphi_{yBc} \\ \varphi_{xB1} \\ \varphi_{xB2} \\ \varepsilon_{zBc} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix} = [f_{mW}] \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix}$$

$$\begin{Bmatrix} s_{xc} \\ s_{y1} \\ s_{y2} \end{Bmatrix} = \begin{bmatrix} 1/k_{sc} & & \\ & 1/k_{s1} & \\ & & 1/k_{s2} \end{bmatrix} \begin{Bmatrix} \mathcal{Q}'_{xc} \\ \mathcal{Q}'_{y1} \\ \mathcal{Q}'_{y2} \end{Bmatrix} = \begin{bmatrix} 1/k_{sc} & & \\ & 1/k_{s1} & \\ & & 1/k_{s2} \end{bmatrix} \begin{bmatrix} 1/l' & 0 & 1/l' & 0 \\ 1/l' & 0 & 1/l' & 0 \\ 1/l' & 0 & 1/l' & 0 \end{bmatrix} \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix}$$

$$\begin{Bmatrix} \varphi_{yAc} \\ \varphi_{xA1} \\ \varphi_{xA2} \\ \varepsilon_{zAc} \\ \varphi_{yBc} \\ \varphi_{xB1} \\ \varphi_{xB2} \\ \varepsilon_{zBc} \\ s_{xc} \\ s_{y1} \\ s_{y2} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & & & 0 & & & & & & & \\ & & & & & & & & & & \\ & & 0 & & [f_{pA}] & & & & & & \\ & & & & & & & & & & \\ \frac{1}{k_{sc}l'} & & & 0 & \frac{1}{k_{sc}l'} & & & 0 & & & \\ & \frac{1}{k_{s1}l'} & & 0 & \frac{1}{k_{s1}l'} & & & 0 & & & \\ & & \frac{1}{k_{s2}l'} & 0 & \frac{1}{k_{s2}l'} & & & 0 & & & \end{bmatrix} \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix} = [f_{pW}] \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix}$$

(2-3-29)

Furthermore, in the same way as Equation (2-3-8),

$$\left\{ \begin{matrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{matrix} \right\} = [n_p]^T \left\{ \begin{matrix} M'_{yAc} \\ M'_{yBc} \\ M'_{xA1} \\ M'_{xB1} \\ M'_{xA2} \\ M'_{xB2} \\ N'_{zc} \end{matrix} \right\} \quad (2-3-30)$$

Therefore, the nonlinear spring displacement vector is obtained from the element face displacement as,

$$\left\{ \begin{matrix} \varphi_{yAc} \\ \varphi_{xA1} \\ \varphi_{xA2} \\ \varepsilon_{zAc} \\ \varphi_{yBc} \\ \varphi_{xB1} \\ \varphi_{xB2} \\ \varepsilon_{zBc} \\ \eta_{yc} \\ \eta_{x1} \\ \eta_{x2} \end{matrix} \right\} = [f_{pW}][n_p]^T \left\{ \begin{matrix} M'_{yAc} \\ M'_{yBc} \\ M'_{xA1} \\ M'_{xB1} \\ M'_{xA2} \\ M'_{xB2} \\ N'_{zc} \end{matrix} \right\} \quad (2-3-31)$$

### In case of direct input wall

Direct input wall model is defined as a line element with a nonlinear shear spring and a nonlinear bending spring in the middle of the element as shown in Figure 2-3-1.

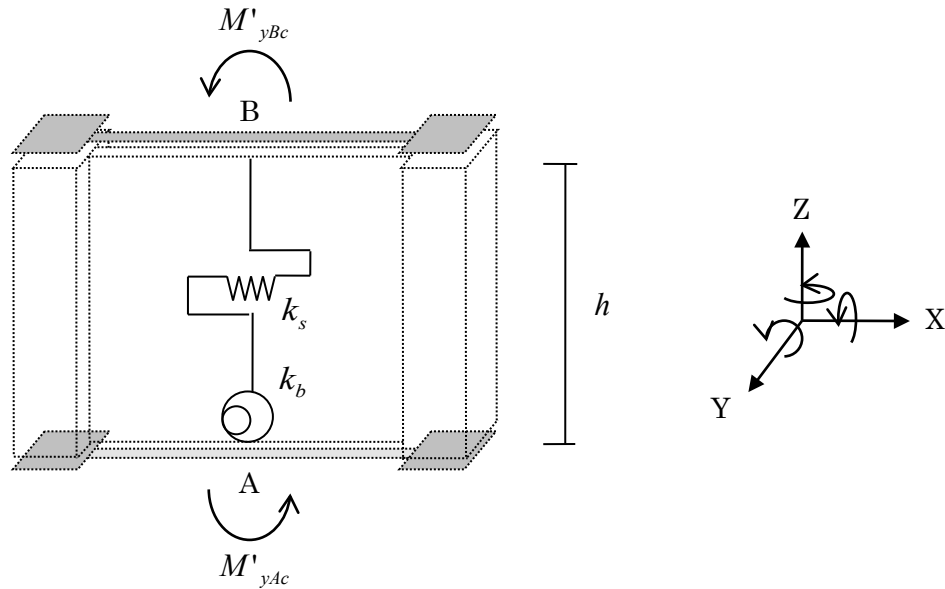


Figure 2-3-5 Element model for wall

This model can be used as an alternative model so called the lumped mass model representing the restoring force characteristics of each layer in the analysis of high-rise building as shown below. The detail of the model is described in Chapter 7.1

## Flexural-shear lumped mass model

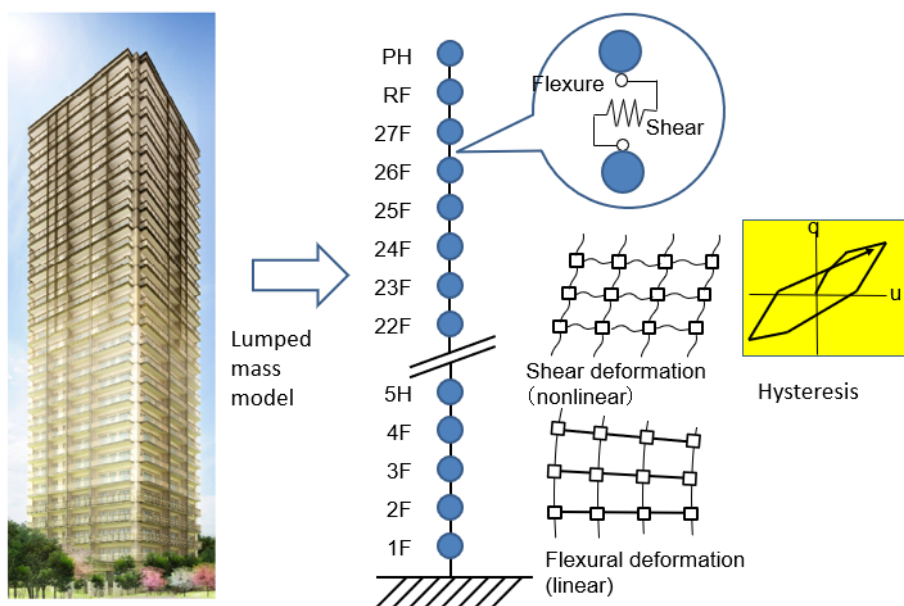


Figure 2-3-6 Lumped mass model of high-rise building

STERA\_3D adopts the formulation to have nonlinear shear and bending springs of the element.

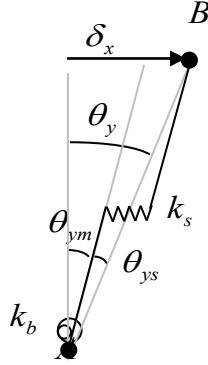


Figure 2-3-7 Nonlinear bending and shear springs

### Force-displacement relationship

The story drift angle,  $\theta_y$ , is composed of the shear component,  $\theta_{y's}$ , and the bending component,  $\theta_{y'm}$ .

$$\theta_y = \frac{\delta_x}{h} = \theta_{y's} + \theta_{y'm} = \frac{\delta_{xs}}{h} + \theta_{y'm} \quad (2-3-32)$$

where,  $\delta_x$  is the story drift and  $\delta_{xs}$  is its shear component. In a matrix form

$$\delta_x = \begin{bmatrix} 1 & h \end{bmatrix} \begin{pmatrix} \delta_{xs} \\ \theta_{y'm} \end{pmatrix} \quad (2-3-33)$$

The nonlinear shear spring is defined as

$$Q_x = k_s \delta_{xs} \quad (2-3-34)$$

The nonlinear bending spring is defined as

$$M_y = k_b \theta_{y'm} \quad (2-3-35)$$

By considering the relationship  $Q_x = \frac{M_y}{h}$ , the force vector of the element is

$$\begin{pmatrix} Q_x \\ M_y \end{pmatrix} = \begin{bmatrix} 1 \\ h \end{bmatrix} Q_x \quad (2-3-36)$$

Therefore, the relationship between the story drift and the shear force is expressed as follows:

$$\delta_x = \begin{bmatrix} 1 & h \end{bmatrix} \begin{pmatrix} \delta_{xs} \\ \theta_{y'm} \end{pmatrix} = \begin{bmatrix} 1 & h \end{bmatrix} \begin{bmatrix} 1/k_s & 0 \\ 0 & 1/k_b \end{bmatrix} \begin{pmatrix} Q_x \\ M_y \end{pmatrix} = \begin{bmatrix} 1/k_s & h/k_b \end{bmatrix} \begin{bmatrix} 1 \\ h \end{bmatrix} Q_x = \left( \frac{1}{k_s} + \frac{h^2}{k_b} \right) Q_x$$

$$Q_x = k_x \delta_x, \quad k_x = \frac{1}{\left( \frac{1}{k_s} + \frac{h^2}{k_b} \right)} \quad (2-3-37)$$

### Including node movement

The relationship between the shear spring displacement and nodal displacement is,

From nodal displacement,

$$\delta_x = u_{xB} - u_{xA} \quad (2-3-38)$$

In a matrix form

$$\delta_x = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \end{Bmatrix} = [\Lambda_L] \begin{Bmatrix} u_{xA} \\ u_{xB} \end{Bmatrix} \quad (2-3-39)$$

### From global node displacement to element node displacement

$$\begin{Bmatrix} u_{xA} \\ u_{xB} \end{Bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{Bmatrix} u_{xA1} \\ u_{xB1} \end{Bmatrix} = [D_W] \begin{Bmatrix} u_{xA1} \\ u_{xB1} \end{Bmatrix} \quad (2-3-40)$$

Transformation from the global node displacements to the element node displacements is;

$$\begin{Bmatrix} u_{xA1} \\ u_{xB1} \end{Bmatrix} = [T_{ixL}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-41)$$

The component of the transformation matrix,  $[T_{ixL}]$ , is discussed in Chapter 4 (Freedom Vector).

### From global node displacement to element face displacement

Transformation from the global node displacement to the spring displacement is,

$$\delta_x = [\Lambda_L][D_W][T_{ixL}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{xL}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-42)$$

### Constitutive equation

Finally, the constitutive equation of the lumped mass model is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{xL}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-3-43)$$

where,

$$[K_{xL}] = [T_{xL}]^T [k_L] [T_{xL}] \quad (2-3-44)$$

**Transformation matrix for nonlinear spring displacement**

The nonlinear spring displacement vector is obtained from the element face displacement as,

$$\begin{pmatrix} \delta_{xs} \\ \theta_m \end{pmatrix} = \begin{bmatrix} 1/k_s & 0 \\ 0 & 1/k_b \end{bmatrix} \begin{pmatrix} Q_x \\ M_y \end{pmatrix} = \begin{bmatrix} 1/k_s & 0 \\ 0 & h/k_b \end{bmatrix} Q_x \quad (2-3-45)$$

## 2.4 Brace

Element model for Brace is defined as a truss element with a nonlinear axial spring and pin-supported at both ends as shown in Figure 2-6-1.

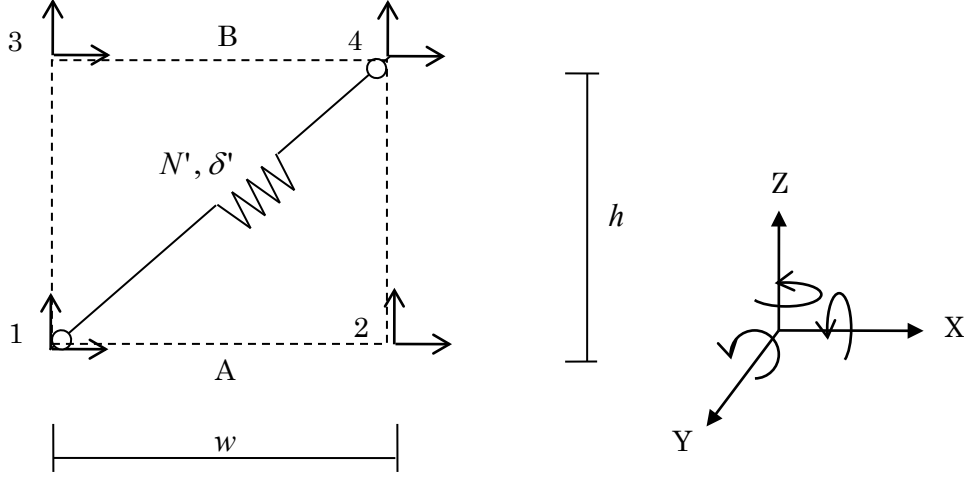


Figure 2-4-1 Element model for brace

### Force-displacement relationship

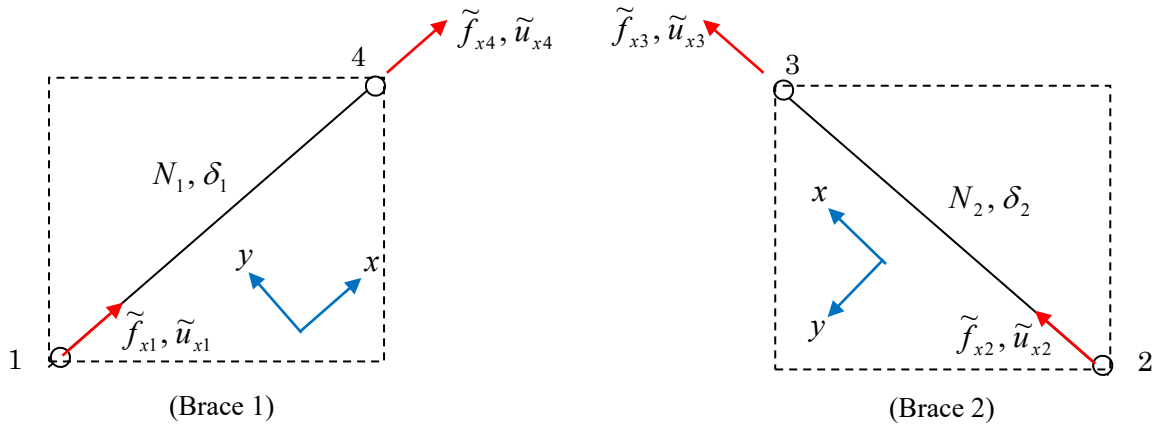


Figure 2-4-2 Local coordinate

The relationship between axial deformation and axial force of the truss element is,

$$N_1 = k_1 \delta_1 \quad (2-4-1)$$

$$N_2 = k_2 \delta_2 \quad (2-4-2)$$

Replacing with the nodal force and displacement in local coordinate along the element,

$$N_1 = -\tilde{f}_{1x} = \tilde{f}_{4x}, \quad \delta_1 = \tilde{u}_{4x} - \tilde{u}_{1x} \quad (2-4-3)$$

$$N_2 = -\tilde{f}_{2x} = \tilde{f}_{3x}, \quad \delta_2 = \tilde{u}_{3x} - \tilde{u}_{2x} \quad (2-4-4)$$

In a matrix form,

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \tilde{u}_{1x} \\ \tilde{u}_{2x} \\ \tilde{u}_{3x} \\ \tilde{u}_{4x} \end{Bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \tilde{u}_{1x} \\ \tilde{u}_{1y} \\ \tilde{u}_{2x} \\ \tilde{u}_{2y} \\ \tilde{u}_{3x} \\ \tilde{u}_{3y} \\ \tilde{u}_{4x} \\ \tilde{u}_{4y} \end{Bmatrix} = [n_b] \{\tilde{u}\} \quad (2-4-5)$$

$$\begin{Bmatrix} \tilde{f}_{1x} \\ \tilde{f}_{2x} \\ \tilde{f}_{3x} \\ \tilde{f}_{4x} \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \rightarrow \{\tilde{f}\} = \begin{Bmatrix} \tilde{f}_{x1} \\ \tilde{f}_{z1} \\ \tilde{f}_{x2} \\ \tilde{f}_{z2} \\ \tilde{f}_{x3} \\ \tilde{f}_{z3} \\ \tilde{f}_{x4} \\ \tilde{f}_{z4} \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = [n_b]^T \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \quad (2-4-6)$$

From Figure 2-4-3, the relation between the nodal forces in local coordinate and those of global coordinate is,

$$\begin{aligned} \tilde{f}_{x1} &= f_{x1} \cos \theta + f_{z1} \sin \theta \\ \tilde{f}_{y1} &= -f_{x1} \sin \theta + f_{z1} \cos \theta \end{aligned} \quad \text{for Brace 1} \quad (2-4-7)$$

and

$$\begin{aligned} \tilde{f}_{x2} &= -f_{x2} \cos \theta + f_{z2} \sin \theta \\ \tilde{f}_{y2} &= -f_{x2} \sin \theta - f_{z2} \cos \theta \end{aligned} \quad \text{for Brace 2} \quad (2-4-8)$$

Eq. (2-4-8) can be also obtained from the Eq. (2-4-7) by replacing  $\theta$  by  $(\pi - \theta)$  and using the formulas  $\sin(\pi - \theta) = \sin \theta$ ,  $\cos(\pi - \theta) = -\cos \theta$ .

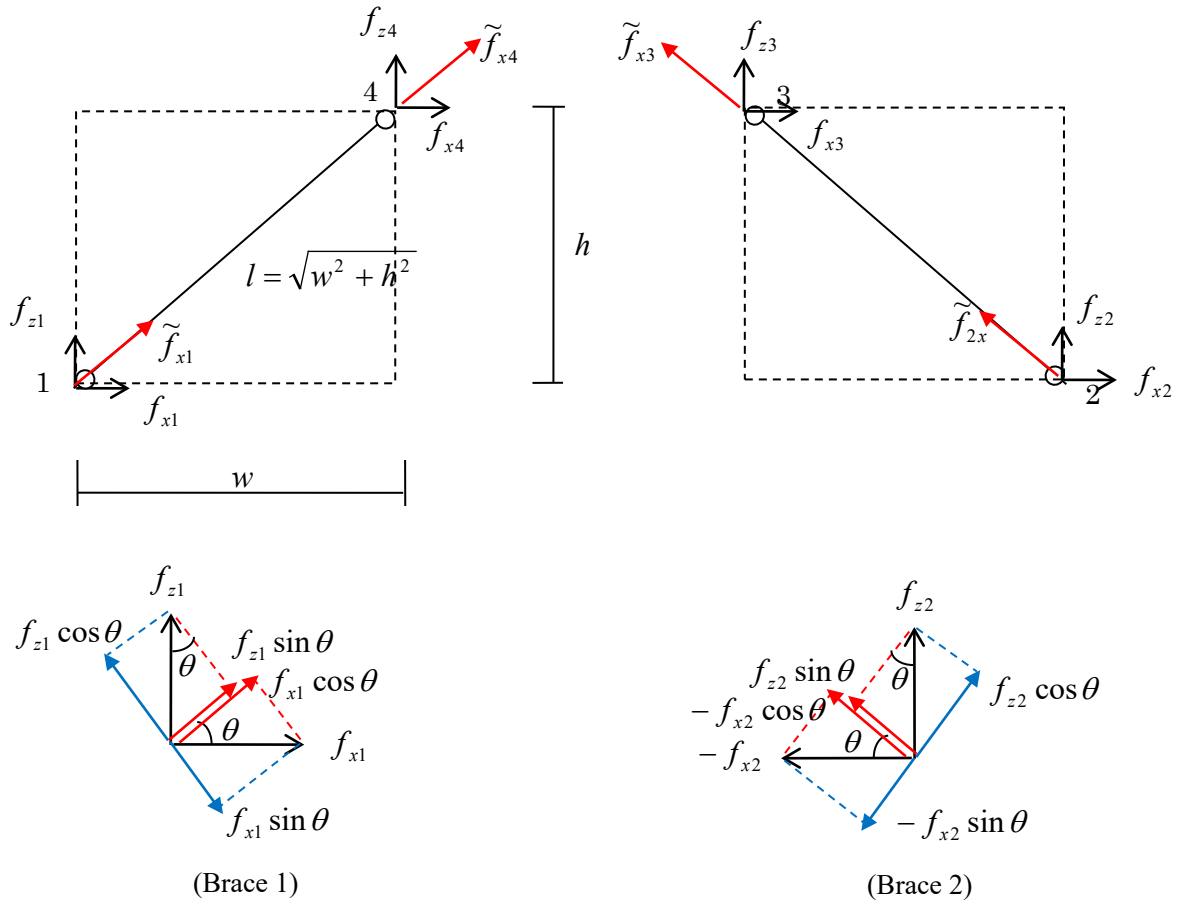


Figure 2-4-3 Coordinate transformation

In a matrix form,

$$\begin{Bmatrix} \tilde{f}_{x1} \\ \tilde{f}_{z1} \\ \tilde{f}_{x2} \\ \tilde{f}_{z2} \\ \tilde{f}_{x3} \\ \tilde{f}_{z3} \\ \tilde{f}_{x4} \\ \tilde{f}_{z4} \end{Bmatrix}_{Local} = \begin{bmatrix} c & s & & & & & \\ -s & c & & & & & \\ & & -c & s & & & \\ & & -s & -c & & & \\ & & & & -c & s & \\ & & & & -s & -c & \\ & & & & & & c & s \\ & & & & & & -s & c \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{z1} \\ f_{x2} \\ f_{z2} \\ f_{x3} \\ f_{z3} \\ f_{x4} \\ f_{z4} \end{Bmatrix}_{Global} = [C_b] \begin{Bmatrix} f_{x1} \\ f_{z1} \\ f_{x2} \\ f_{z2} \\ f_{x3} \\ f_{z3} \\ f_{x4} \\ f_{z4} \end{Bmatrix} \quad (2-4-9)$$

where

$$c = \cos \theta = \frac{w}{l}, \quad s = \sin \theta = \frac{h}{l}$$

Since  $[C_b][C_b]^T = I$ ,  $[C_b]$  is an orthogonal matrix, therefore,

$$[C_b]^{-1} = [C_b]^T \quad (2-4-10)$$

In a similar manner, from Figure 2-4-4, the relation between the nodal displacements in local coordinate and those of global coordinate can be obtained as,

$$\begin{aligned} u_{x1} &= \tilde{u}_{x1} \cos \theta - \tilde{u}_{z1} \sin \theta \\ u_{z1} &= \tilde{u}_{x1} \sin \theta + \tilde{u}_{z1} \cos \theta \end{aligned} \quad \text{for Brace 1} \quad (2-4-11)$$

and

$$\begin{aligned} u_{x2} &= -\tilde{u}_{x2} \cos \theta - \tilde{u}_{z2} \sin \theta \\ u_{z2} &= \tilde{u}_{x2} \sin \theta - \tilde{u}_{z2} \cos \theta \end{aligned} \quad \text{for Brace 2} \quad (2-4-12)$$

Eq. (2-4-12) can be also obtained from the Eq. (2-4-11) by replacing  $\theta$  by  $(\pi - \theta)$  and using the formulas  $\sin(\pi - \theta) = \sin \theta$ ,  $\cos(\pi - \theta) = -\cos \theta$ .

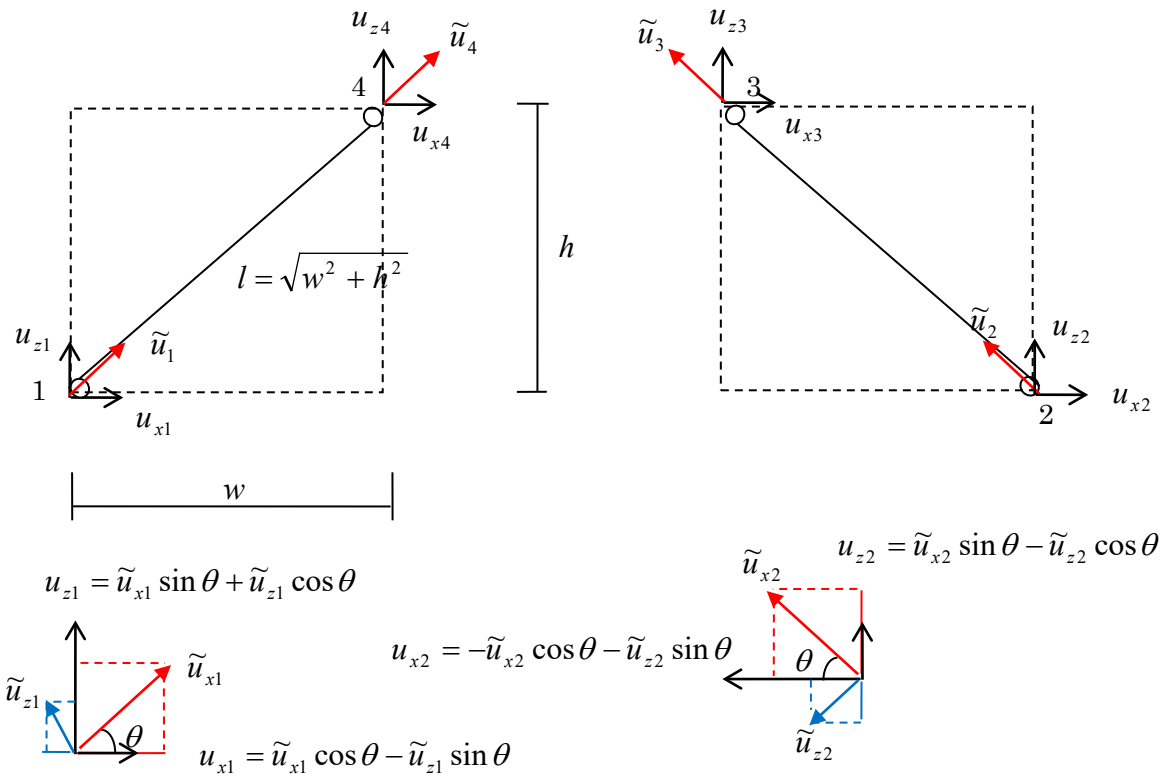


Figure 2-4-4 Coordinate transformation

In a matrix form,

$$\begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{Bmatrix}_{Local} = \begin{bmatrix} c & -s & & & & & \\ & s & c & & & & \\ & & & -c & -s & & \\ & & & s & -c & & \\ & & & & & -c & -s \\ & & & & & s & -c \\ & & & & & & c & -s \\ & & & & & & s & c \end{bmatrix} \begin{Bmatrix} \tilde{u}_{x1} \\ \tilde{u}_{z1} \\ \tilde{u}_{x2} \\ \tilde{u}_{z2} \\ \tilde{u}_{x3} \\ \tilde{u}_{z3} \\ \tilde{u}_{x4} \\ \tilde{u}_{z4} \end{Bmatrix}_{Global} = [C_b]^T \begin{Bmatrix} \tilde{u}_{x1} \\ \tilde{u}_{z1} \\ \tilde{u}_{x2} \\ \tilde{u}_{z2} \\ \tilde{u}_{x3} \\ \tilde{u}_{z3} \\ \tilde{u}_{x4} \\ \tilde{u}_{z4} \end{Bmatrix}$$

The stiffness matrix of brace element is,

$$\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \quad \text{or} \quad \{N\} = [\tilde{k}] \{\delta\} \quad (2-4-13)$$

Where

$$\{\delta\} = [n_b] \{\tilde{u}\} = [n_b] \left( [C_b]^T \right)^{-1} \{u\} = [n_b] [C_b] \{u\} \quad (2-4-14)$$

$$\{f\} = [C_b]^{-1} \{\tilde{f}\} = [C_b]^T [n_b]^T \{N\} \quad (2-4-15)$$

### From global node displacement to element node displacement

Transformation from the global node displacement to the element node displacement is;

$$\{u\} = [T_{ixBr}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-4-16)$$

The component of the transformation matrix,  $[T_{ixBr}]$ , is discussed in Chapter 4 (Freedom Vector).

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\{\delta\} = [n_b] [C_b] \{u\} = [n_b] [C_b] [T_{ixBr}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{xBr}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-4-17)$$

### Constitutive equation

Finally, the constitutive equation of the brace is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{xBr}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-4-18)$$

where,

$$[K_{xBr}] = [T_{xBr}]^T [k_{Br}] [T_{xBr}] \quad (2-4-19)$$

**In case of Y-direction brace**

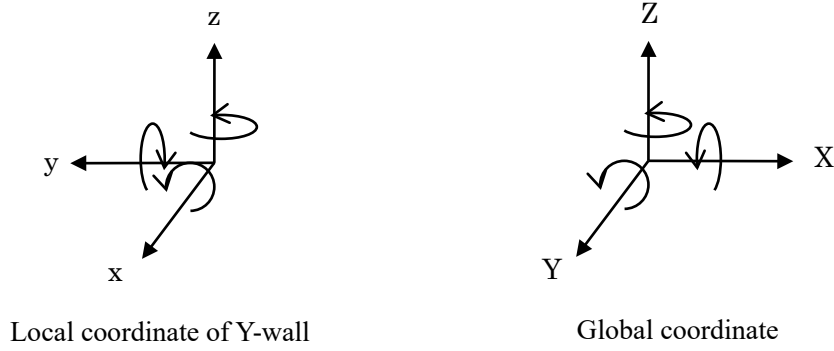


Figure 2-7-2 Relation between local coordinate and global coordinate

In case of Y-direction brace, transformation of the sign of the vector components of the element coordinate is,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{Y-Beam} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_{Global} \quad (2-4-20)$$

Therefore

$$\begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{Bmatrix}_{Y-Brace} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ u_{z1} \\ u_{y2} \\ u_{z2} \\ u_{y3} \\ u_{z3} \\ u_{y4} \\ u_{z4} \end{Bmatrix}_{Global} = \begin{Bmatrix} u_{y1} \\ u_{z1} \\ u_{y2} \\ u_{z2} \\ u_{y3} \\ u_{z3} \\ u_{y4} \\ u_{z4} \end{Bmatrix}_{Global} \quad (2-4-21)$$

Transformation from the global node displacement to the element node displacement is;

$$\{u\} = [T_{iyBr}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-4-22)$$

Transformation from the global node displacement to the element face displacement is,

$$\{\delta\} = [T_{yBr}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix}, \quad [T_{yBr}] = [n_b][C_b][T_{iyBr}] \quad (2-4-23)$$

### Constitutive equation

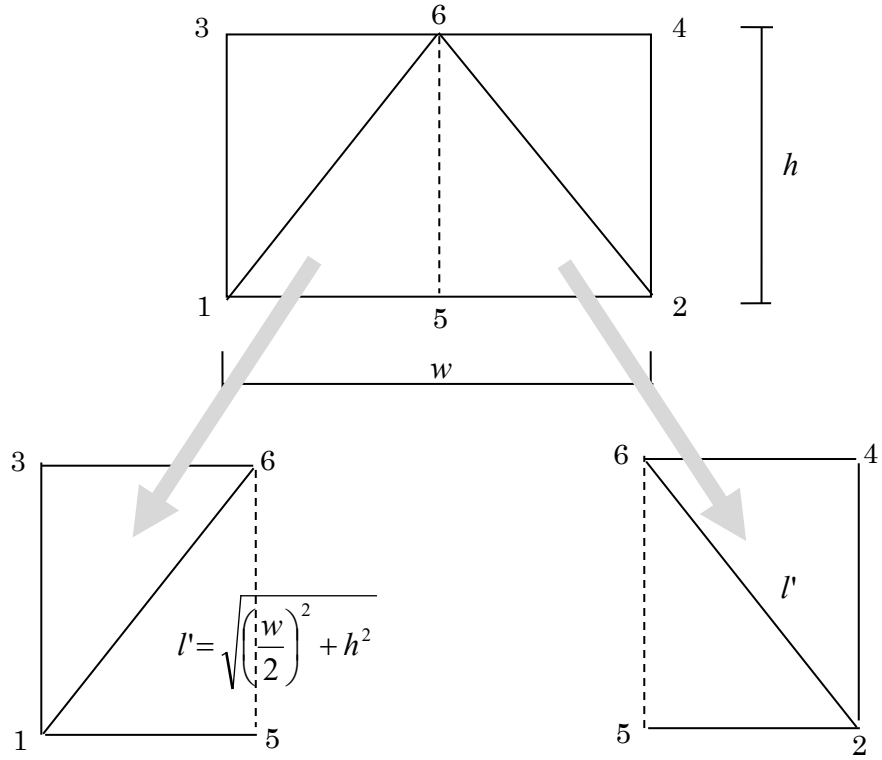
The constitutive equation of the Y-direction brace is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{yBr}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-4-24)$$

where,

$$[K_{yBr}] = [T_{yBr}]^T [k_{Br}] [T_{yBr}] \quad (2-4-25)$$

**In case of K-brace (or Cheveron brace)**



For the left half part, as we defined before for the ordinary brace, the stiffness equation of brace element is,

$$\{f\}_L = [k]_L \{u\}_L \quad [k]_L = [C_b]^T [n_b]^T [\tilde{k}]_L [n_b] [C_b] \quad (2-4-26)$$

where

$$\{f\}_L = \{f_{x1} \quad f_{z1} \quad f_{x5} \quad f_{z5} \quad f_{x3} \quad f_{z3} \quad f_{x6} \quad f_{z6}\}^T$$

$$\{u\}_L = \{u_{x1} \quad u_{z1} \quad u_{x5} \quad u_{z5} \quad u_{x3} \quad u_{z3} \quad u_{x6} \quad u_{z6}\}^T$$

$$[\tilde{k}]_L = \begin{bmatrix} k_{1,L} & 0 \\ 0 & k_{2,L} \end{bmatrix}, \quad [n_b] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$[C_b] = \begin{bmatrix} c & s & & & & & \\ -s & c & & & & & \\ & & -c & s & & & \\ & & -s & -c & & & \\ & & & & -c & s & \\ & & & & -s & -c & \\ & & & & & & c & s \\ & & & & & & -s & c \end{bmatrix}, \quad c = \cos \theta = \frac{(w/2)}{l'}, \quad s = \sin \theta = \frac{h}{l'}$$

For the right half part, in the same way, the stiffness equation of brace element is,

$$\{f\}_R = [k]_R \{u\}_R \quad [k]_R = [C_b]^T [n_b]^T [\tilde{k}]_R [n_b] [C_b] \quad (2-4-27)$$

where

$$\{f\}_R = \{f_{x5} \quad f_{z5} \quad f_{x2} \quad f_{z2} \quad f_{x6} \quad f_{z6} \quad f_{x4} \quad f_{z4}\}^T$$

$$\{u\}_R = \{u_{x5} \quad u_{z5} \quad u_{x2} \quad u_{z2} \quad u_{x6} \quad u_{z6} \quad u_{x4} \quad u_{z4}\}^T$$

We can express the nodal displacement vector as,

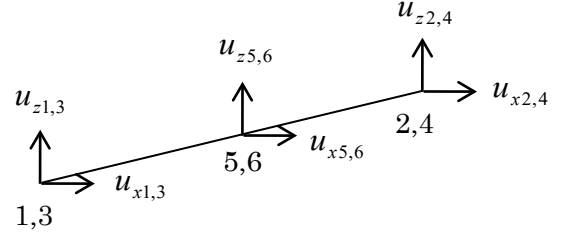
$$\{u\}_L = \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x5} \\ u_{z5} \\ u_{x3} \\ u_{z3} \\ u_{x6} \\ u_{z6} \end{Bmatrix} = \begin{bmatrix} [1] & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \\ u_{x5} \\ u_{z5} \\ u_{x6} \\ u_{z6} \end{Bmatrix} = [D_L] \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \\ u_{x5} \\ u_{z5} \\ u_{x6} \\ u_{z6} \end{Bmatrix}$$

$$\{u\}_R = \begin{Bmatrix} u_{x5} \\ u_{z5} \\ u_{x2} \\ u_{z2} \\ u_{x6} \\ u_{z6} \\ u_{x4} \\ u_{z4} \end{Bmatrix} = \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \\ u_{x5} \\ u_{z5} \\ u_{x6} \\ u_{z6} \end{Bmatrix} = [D_R] \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \\ u_{x5} \\ u_{z5} \\ u_{x6} \\ u_{z6} \end{Bmatrix}$$

We assume the displacements of intermediate nodes, 5 and 6, are calculated from those of end nodes as follows,

$$u_{x5} = \frac{1}{2}(u_{x1} + u_{x2}), \quad u_{z5} = \frac{1}{2}(u_{z1} + u_{z2})$$

$$u_{x6} = \frac{1}{2}(u_{x3} + u_{x4}), \quad u_{z6} = \frac{1}{2}(u_{z3} + u_{z4})$$



In a matrix form

$$\left\{ \begin{matrix} u_{x5} \\ u_{z5} \\ u_{x6} \\ u_{z6} \end{matrix} \right\}_{Local} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{bmatrix} \left\{ \begin{matrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{matrix} \right\}_{Local} = [h_{Ch}] \left\{ \begin{matrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{matrix} \right\}_{Local} \quad (2-4-28)$$

Therefore,

$$\left\{ \begin{matrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \\ \hline u_{x5} \\ u_{z5} \\ u_{x6} \\ u_{z6} \end{matrix} \right\} = \begin{bmatrix} [I] \\ [h_{Ch}] \end{bmatrix} \left\{ \begin{matrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{matrix} \right\}_{Local} = [T_{Ch}] \left\{ \begin{matrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{matrix} \right\}_{Local} \quad (2-4-29)$$

Therefore,

$$\{u\}_L = \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x5} \\ u_{z5} \\ u_{x3} \\ u_{z3} \\ u_{x6} \\ u_{z6} \end{Bmatrix} = [D_L] \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \\ u_{x5} \\ u_{z5} \\ u_{x6} \\ u_{z6} \end{Bmatrix} = [D_L][T_{Ch}] \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{Bmatrix}_{Local}, \quad \{u\}_R = \begin{Bmatrix} u_{x5} \\ u_{z5} \\ u_{x2} \\ u_{z2} \\ u_{x6} \\ u_{z6} \\ u_{x4} \\ u_{z4} \end{Bmatrix} = [D_R] \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \\ u_{x5} \\ u_{z5} \\ u_{x6} \\ u_{z6} \end{Bmatrix} = [D_R][T_{Ch}] \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{Bmatrix}_{Local} \quad (2-4-30)$$

Finally the force-displacement relationship of Chevron brace is,

$$[k]_L = [C_b]^T [n_b]^T [\tilde{k}]_L [n_b][C_b] \quad (2-4-31)$$

$$\begin{Bmatrix} f_{x1} \\ f_{z1} \\ f_{x2} \\ f_{z2} \\ f_{x3} \\ f_{z3} \\ f_{x4} \\ f_{z4} \end{Bmatrix}_{Local} = \left( [T_{Ch}]^T [D_L]^T [k]_L [D_L][T_{Ch}] + [T_{Ch}]^T [D_R]^T [k]_R [D_R][T_{Ch}] \right) \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \\ u_{x3} \\ u_{z3} \\ u_{x4} \\ u_{z4} \end{Bmatrix} \quad (2-4-32)$$

## 2.5 External Spring

### 1) Axial spring

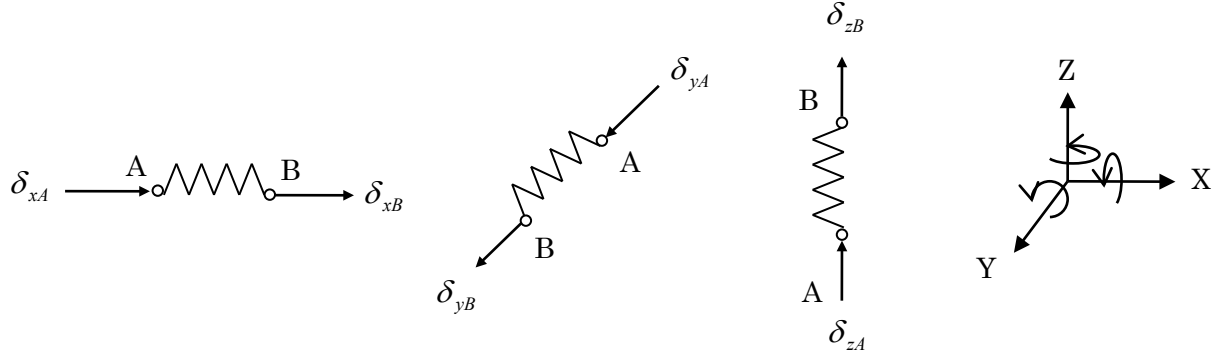


Figure 2-5-1 Element model for external spring

#### Force-displacement relationship for the element

The relationship between the displacement vector and force vector of the elastic element in Figure 2-5-1 is expressed as follows:

$$\{N'_i\} = [k_E] \{\delta'_i\}, \quad i = x, y, z \quad (2-5-1)$$

$$\delta'_x = u_{xB} - u_{xA}$$

$$\delta'_y = u_{yB} - u_{yA} \quad (2-5-2)$$

$$\delta'_z = \delta_{zB} - \delta_{zA}$$

Therefore

$$\{\delta'_x\} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} = [n_{xE}] \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} \quad (2-5-3)$$

$$\{\delta'_y\} = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} = [n_{yE}] \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} \quad (2-5-4)$$

$$\{\delta'_z\} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} = [n_{zE}] \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} \quad (2-5-5)$$

#### From global node displacement to element node displacement

$$\begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} = [T_{iE}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-5-6)$$

The component of the transformation matrix,  $[T_{iE}]$ , is discussed in Chapter 4 (Freedom Vector).

#### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\{\delta'_i\} = [n_{iE}] [T_{iE}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_E] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix}, \quad i = x, y, z \quad (2-5-7)$$

#### Constitutive equation

The constitutive equation of the external spring is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_E] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-5-8)$$

where,

$$[K_E] = [T_E]^T [k_E] [T_E] \quad (2-5-9)$$

## 2) Rotational spring

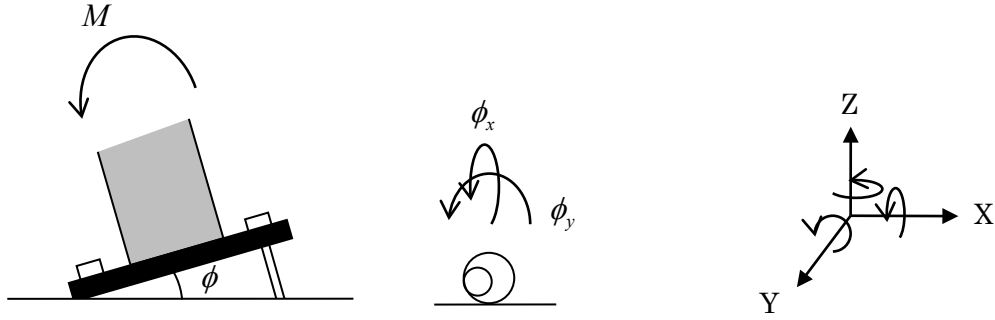


Figure 2-5-2 Element model for external spring

### Force-displacement relationship for the element

The relationship between the displacement vector and force vector of the elastic element in Figure 2-5-2 is expressed as follows:

$$\begin{Bmatrix} M_y \\ M_x \end{Bmatrix} = \begin{bmatrix} k_{ry} & 0 \\ 0 & k_{rx} \end{bmatrix} \begin{Bmatrix} \phi_y \\ \phi_x \end{Bmatrix} = [k_{rE}] \{\theta'\} \quad (2-5-10)$$

$$\begin{aligned} \phi_y &= \theta_{yB} - \theta_{yA} \\ \phi_x &= \theta_{xB} - \theta_{xA} \end{aligned} \quad (2-5-11)$$

Therefore

$$\begin{Bmatrix} \phi_y \\ \phi_x \end{Bmatrix} = \begin{bmatrix} -1 & 1 & & \\ & & -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{yA} \\ \theta_{yB} \\ \theta_{xA} \\ \theta_{xB} \end{Bmatrix} = [n_{rE}] \begin{Bmatrix} \theta_{yA} \\ \theta_{yB} \\ \theta_{xA} \\ \theta_{xB} \end{Bmatrix} \quad (2-5-12)$$

### From global node displacement to element node displacement

$$\begin{Bmatrix} \theta_{yA} \\ \theta_{yB} \\ \theta_{xA} \\ \theta_{xB} \end{Bmatrix} = [T_{rE}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-5-13)$$

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} = [n_{rE}] [T_{rE}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_E] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-5-14)$$

### Constitutive equation

The constitutive equation of the external spring is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_E] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-5-15)$$

where,

$$[K_E] = [T_E]^T [k_E] [T_E] \quad (2-5-16)$$

### 3) Pendulum element

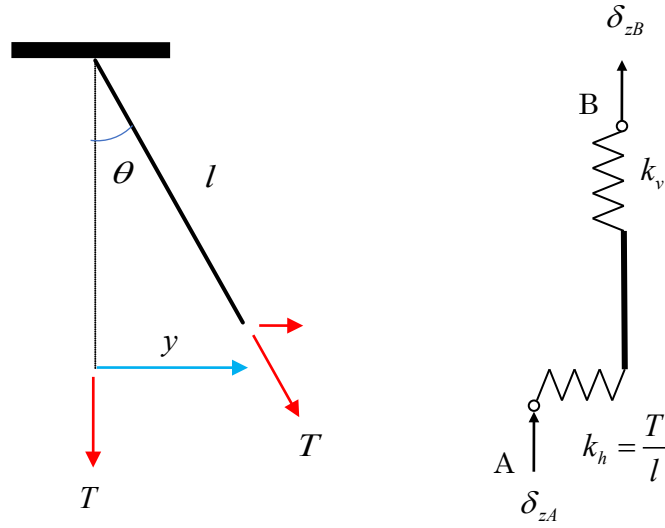


Figure 2-5-3 Element model for pendulum element

#### Force-displacement relationship for the element

The relationship between the displacement vector and force vector of the elastic element in Figure 2-5-3 is expressed as follows:

$$\begin{Bmatrix} Q'_x \\ Q'_y \\ N'_z \end{Bmatrix} = \begin{bmatrix} k_h & & \\ & k_h & \\ & & k_v \end{bmatrix} \begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix} = [k_{pE}] \begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix} \quad (2-5-17)$$

From node displacements, relative displacements are;

$$\begin{aligned} \delta'_x &= u_{xB} - u_{xA} \\ \delta'_y &= u_{yB} - u_{yA} \\ \delta'_z &= \delta_{zB} - \delta_{zA} \end{aligned} \quad (2-5-18)$$

Therefore

$$\begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix} = \begin{bmatrix} -1 & 1 & & & \\ & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} = [n_{pE}] \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} \quad (2-5-19)$$

### From global node displacement to element node displacement

Transformation from the global node displacement to the element node displacement is,

$$\begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} = [T_{pE}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-5-20)$$

The component of the transformation matrix,  $[T_{iBl}]$ , is discussed in Chapter 4 (Freedom Vector).

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix} = [n_{pE}] [T_{pE}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_E] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-5-21)$$

### Constitutive equation

The constitutive equation of the Base isolation is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_E] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-5-22)$$

where,

$$[K_E] = [T_E]^T [k_E] [T_E] \quad (2-5-23)$$

## 2.6 Base Isolation

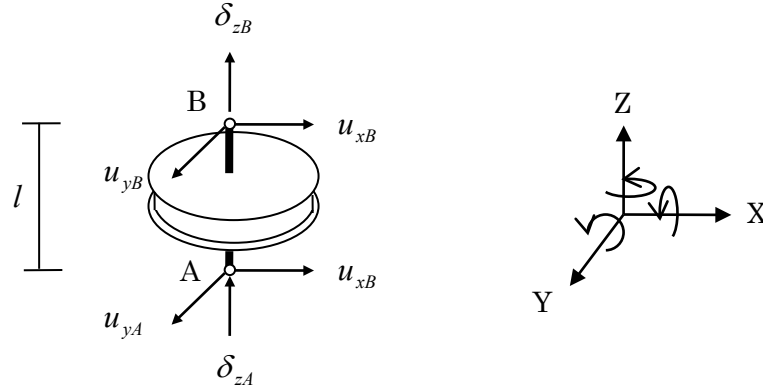


Figure 2-6-1 Element model for base isolation

### Force-displacement relationship for the element

The relationship between the displacement vector and force vector of the element is expressed as follows:

$$\begin{Bmatrix} Q'_x \\ Q'_y \end{Bmatrix} = [k_{pBI}] \begin{Bmatrix} \delta'_x \\ \delta'_y \end{Bmatrix} \quad (2-6-1)$$

Including the axial stiffness,

$$\begin{Bmatrix} Q'_x \\ Q'_y \\ \delta'_z \end{Bmatrix} = \begin{bmatrix} [k_{pBI}] & 0 \\ 0 & \frac{EA}{l'} \end{bmatrix} \begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix} = [k_{BI}] \begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix} \quad (2-6-2)$$

From node displacements, relative displacements are;

$$\begin{aligned} \delta'_x &= u_{xB} - u_{xA} \\ \delta'_y &= u_{yB} - u_{yA} \\ \delta'_z &= \delta_{zB} - \delta_{zA} \end{aligned} \quad (2-6-3)$$

Therefore

$$\begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix} = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} = [n_{BI}] \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} \quad (2-6-4)$$

### From global node displacement to element node displacement

Transformation from the global node displacement to the element node displacement is,

$$\begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{yA} \\ u_{yB} \\ \delta_{zA} \\ \delta_{zB} \end{Bmatrix} = [T_{iBI}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-6-5)$$

The component of the transformation matrix,  $[T_{iBI}]$ , is discussed in Chapter 4 (Freedom Vector).

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \delta'_x \\ \delta'_y \\ \delta'_z \end{Bmatrix} = [n_{BI}] [T_{iBI}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{BI}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-6-6)$$

### Constitutive equation

The constitutive equation of the Base isolation is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{BI}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-6-7)$$

where,

$$[K_{BI}] = [T_{BI}]^T [k_{BI}] [T_{BI}] \quad (2-6-8)$$

## 2.7 Masonry Wall

Element model for Masonry wall is defined as a line element with a nonlinear shear spring and a vertical spring in the middle of the wall panel as shown in Figure 2-6-1.

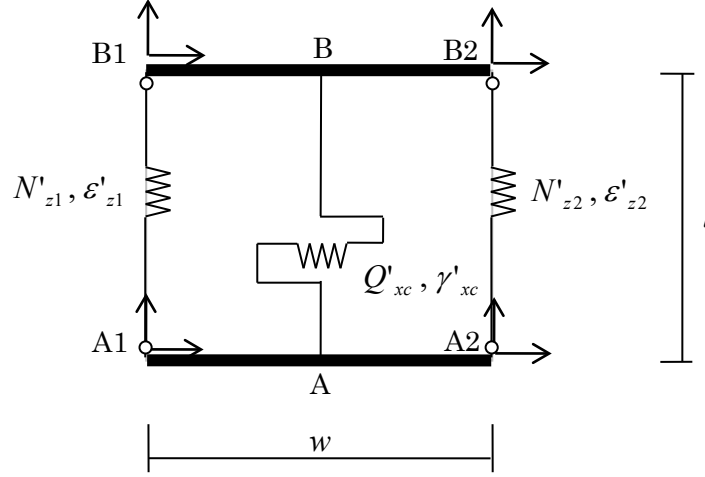


Figure 2-7-1 Element model for masonry wall

### Force-displacement relationship

The relationship between the shear deformation and shear force of the nonlinear shear spring is,

$$Q'_{xc} = k_{sx} \gamma'_{xc} \quad (2-7-1)$$

For axial spring,

$$N'_{z1} = k_z \epsilon'_{z1}, \quad N'_{z2} = k_z \epsilon'_{z2} \quad (2-7-2)$$

In a matrix form,

$$\begin{Bmatrix} Q'_{xc} \\ N'_{z1} \\ N'_{z2} \end{Bmatrix} = \begin{bmatrix} k_{sx} & 0 & 0 \\ 0 & k_z & 0 \\ 0 & 0 & k_z \end{bmatrix} \begin{Bmatrix} \gamma'_{xc} \\ \epsilon'_{z1} \\ \epsilon'_{z2} \end{Bmatrix} = [k_N] \begin{Bmatrix} \gamma'_{xc} \\ \epsilon'_{z1} \\ \epsilon'_{z2} \end{Bmatrix} \quad (2-7-3)$$

### Including node movement

The shear angle of the frame with four nodes, A1, A2, B1, B2, is defined as,

$$\tau = \frac{\partial \delta_z}{\partial x} + \frac{\partial u_x}{\partial z} \quad (2-7-4)$$

where,

$$\frac{\partial \delta_z}{\partial x} \approx \frac{1}{2} \left( \frac{\delta_{zA2} - \delta_{zA1}}{w} + \frac{\delta_{zB2} - \delta_{zB1}}{w} \right) \quad (2-7-5)$$

$$\frac{\partial u_z}{\partial z} \approx \frac{1}{2} \left( \frac{u_{xB1} - u_{xA1}}{l} + \frac{u_{xB2} - u_{xA2}}{l} \right) \quad (2-7-6)$$

The shear deformation,  $\gamma'_{xc}$ , is then,

$$\gamma'_{xc} = \tau l = \frac{l}{2w} (\delta_{zA2} - \delta_{zA1} + \delta_{zB2} - \delta_{zB1}) + \frac{1}{2} (u_{xB1} - u_{xA1} + u_{xB2} - u_{xA2}) \quad (2-7-7)$$

The axial deformation,  $\epsilon'_{z1}, \epsilon'_{z2}$ , is,

$$\epsilon'_{z1} = \delta_{zB1} - \delta_{zA1}, \quad \epsilon'_{z2} = \delta_{zB2} - \delta_{zA2} \quad (2-7-8)$$

In a matrix form,

$$\begin{Bmatrix} \gamma'_{xc} \\ \epsilon'_{z1} \\ \epsilon'_{z2} \end{Bmatrix} = \begin{bmatrix} -0.5 & -0.5 \frac{l}{w} & -0.5 & 0.5 \frac{l}{w} & 0.5 & -0.5 \frac{l}{w} & 0.5 & 0.5 \frac{l}{w} \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ u_{xA2} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ u_{xB2} \\ \delta_{zB2} \end{Bmatrix} = [D_N] \begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ u_{xA2} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ u_{xB2} \\ \delta_{zB2} \end{Bmatrix} \quad (2-7-9)$$

#### From global node displacement to element node displacement

Transformation from the global node displacement to the element node displacement is;

$$\begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ u_{xA2} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ u_{xB2} \\ \delta_{zB2} \end{Bmatrix} = [T_{ixN}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-7-10)$$

The component of the transformation matrix,  $[T_{ixN}]$ , is discussed in Chapter 4 (Freedom Vector).

#### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \gamma'_{xc} \\ \varepsilon'_{z1} \\ \varepsilon'_{z2} \end{Bmatrix} = [D_N] [T_{ixN}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{xN}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-7-11)$$

**In case of Y-direction wall**

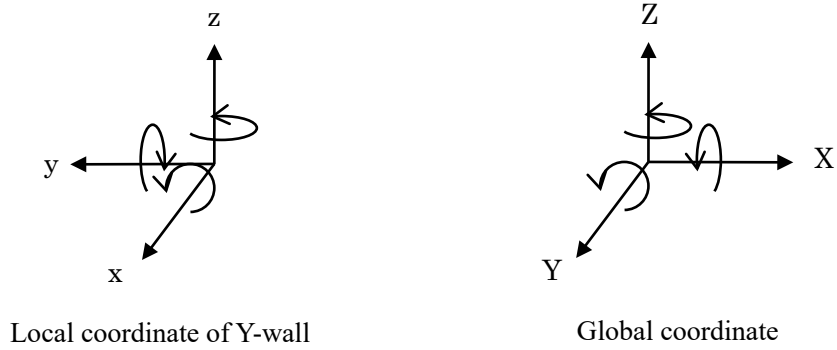


Figure 2-7-2 Relation between local coordinate and global coordinate

In case of Y-direction wall, the wall panel direction coincides to the Y-axis in the global coordinate, transformation of the sign of the vector components of the element coordinate is,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{Y-Beam} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_{Global} \quad (2-7-12)$$

Therefore

$$\begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ u_{xA2} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ u_{xB2} \\ \delta_{zB2} \end{Bmatrix}_{Y-Wall} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ u_{yA2} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ u_{yB2} \\ \delta_{zB2} \end{Bmatrix}_{Global} = \begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ u_{yA2} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ u_{yB2} \\ \delta_{zB2} \end{Bmatrix}_{Global} \quad (2-7-13)$$

Transformation from the global node displacement to the element node displacement is;

$$\begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ u_{yA2} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ u_{yB2} \\ \delta_{zB2} \end{Bmatrix} = [T_{iyN}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-7-14)$$

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \gamma'_{xc} \\ \varepsilon'_{z1} \\ \varepsilon'_{z2} \end{Bmatrix} = [D_N] [T_{iyN}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{yN}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-7-15)$$

### Constitutive equation

Finally, the constitutive equation of the wall is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{xN}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-7-16)$$

where,

$$[K_{xN}] = [T_{xN}]^T [k_N] [T_{xN}] \quad (2-7-17)$$

For Y-wall,

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{yN}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-7-18)$$

where,

$$[K_{yN}] = [T_{yN}]^T [k_N] [T_{yN}] \quad (2-7-19)$$

## 2.8 Passive Damper

Element model for passive damper with a shear spring is defined as a line element with a nonlinear shear spring as shown in Figure 2-8-1.

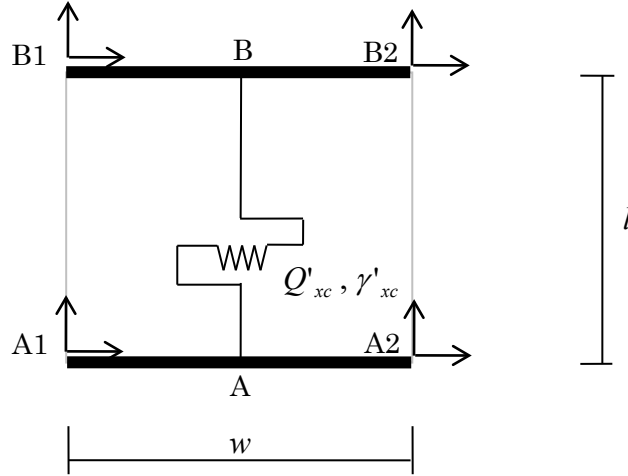


Figure 2-8-1 Element model for passive damper

### Force-displacement relationship

The relationship between the shear deformation and shear force of the nonlinear shear spring is,

$$Q'_{xc} = k_{sx} \gamma'_{xc} \quad (2-8-1)$$

### Including node movement

The shear angle of the frame with four nodes, A1, A2, B1, B2, is defined as,

$$\tau = \frac{\partial \delta_z}{\partial x} + \frac{\partial u_x}{\partial z} \quad (2-8-2)$$

where,

$$\frac{\partial \delta_z}{\partial x} \approx \frac{1}{2} \left( \frac{\delta_{zA2} - \delta_{zA1}}{w} + \frac{\delta_{zB2} - \delta_{zB1}}{w} \right) \quad (2-8-3)$$

$$\frac{\partial u_x}{\partial z} \approx \frac{1}{2} \left( \frac{u_{xB1} - u_{xA1}}{l} + \frac{u_{xB2} - u_{xA2}}{l} \right) \quad (2-8-4)$$

The shear deformation,  $\gamma'_{xc}$ , is then,

$$\gamma'_{xc} = \tau l = \frac{l}{2w} (\delta_{zA2} - \delta_{zA1} + \delta_{zB2} - \delta_{zB1}) + \frac{1}{2} (u_{xB1} - u_{xA1} + u_{xB2} - u_{xA2}) \quad (2-8-5)$$

The axial deformation,  $\varepsilon'_{z1}, \varepsilon'_{z2}$ , is,

$$\varepsilon'_{z1} = \delta_{zB1} - \delta_{zA1}, \quad \varepsilon'_{z2} = \delta_{zB2} - \delta_{zA2} \quad (2-8-6)$$

In a matrix form,

$$\begin{Bmatrix} \gamma'_{xc} \\ \varepsilon'_{z1} \\ \varepsilon'_{z2} \end{Bmatrix} = \begin{bmatrix} -0.5 & -0.5 \frac{l}{w} & -0.5 & 0.5 \frac{l}{w} & 0.5 & -0.5 \frac{l}{w} & 0.5 & 0.5 \frac{l}{w} \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ u_{xA2} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ u_{xB2} \\ \delta_{zB2} \end{Bmatrix} = [D_D] \begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ u_{xA2} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ u_{xB2} \\ \delta_{zB2} \end{Bmatrix} \quad (2-8-7)$$

### From global node displacement to element node displacement

Transformation from the global node displacement to the element node displacement is;

$$\begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ u_{xA2} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ u_{xB2} \\ \delta_{zB2} \end{Bmatrix} = [T_{ixD}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-8-8)$$

The component of the transformation matrix,  $[T_{ixD}]$ , is discussed in Chapter 4 (Freedom Vector).

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \gamma'_{xc} \\ \varepsilon'_{z1} \\ \varepsilon'_{z2} \end{Bmatrix} = [D_D][T_{ixD}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{xD}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-8-9)$$

**In case of Y-direction damper**

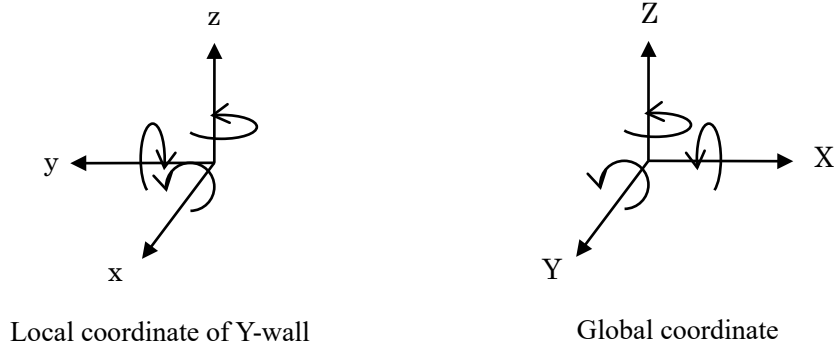


Figure 2-8-2 Relation between local coordinate and global coordinate

In case of Y-direction damper, the damper direction coincides to the Y-axis in the global coordinate, transformation of the sign of the vector components of the element coordinate is,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{Y-Beam} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_{Global} \quad (2-8-10)$$

Therefore

$$\begin{Bmatrix} u_{xA1} \\ \delta_{zA1} \\ u_{xA2} \\ \delta_{zA2} \\ u_{xB1} \\ \delta_{zB1} \\ u_{xB2} \\ \delta_{zB2} \end{Bmatrix}_{Y-Wall} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ u_{yA2} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ u_{yB2} \\ \delta_{zB2} \end{Bmatrix}_{Global} = \begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ u_{yA2} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ u_{yB2} \\ \delta_{zB2} \end{Bmatrix}_{Global} \quad (2-8-11)$$

Transformation from the global node displacement to the element node displacement is;

$$\begin{Bmatrix} u_{yA1} \\ \delta_{zA1} \\ u_{yA2} \\ \delta_{zA2} \\ u_{yB1} \\ \delta_{zB1} \\ u_{yB2} \\ \delta_{zB2} \end{Bmatrix} = [T_{yD}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-8-12)$$

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \gamma'_{xc} \\ \varepsilon'_{z1} \\ \varepsilon'_{z2} \end{Bmatrix} = [D_D] [T_{iyD}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{yD}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-8-13)$$

### Constitutive equation

Finally, the constitutive equation of the damper is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{xD}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-8-14)$$

where,

$$[K_{xD}] = [T_{xD}]^T [k_D] [T_{xD}] \quad (2-8-15)$$

For Y-damper,

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{yD}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-8-16)$$

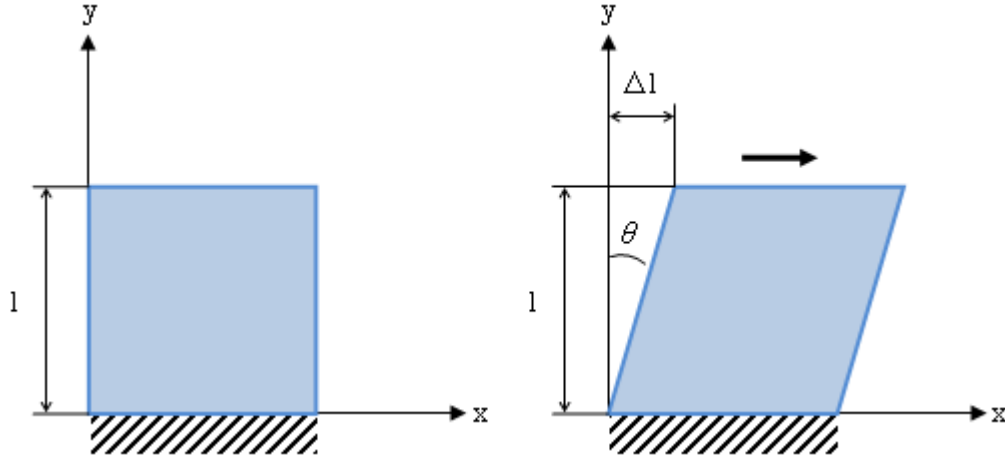
where,

$$[K_{yD}] = [T_{yD}]^T [k_D] [T_{yD}] \quad (2-8-17)$$

## Appendix : Calculation of shear component

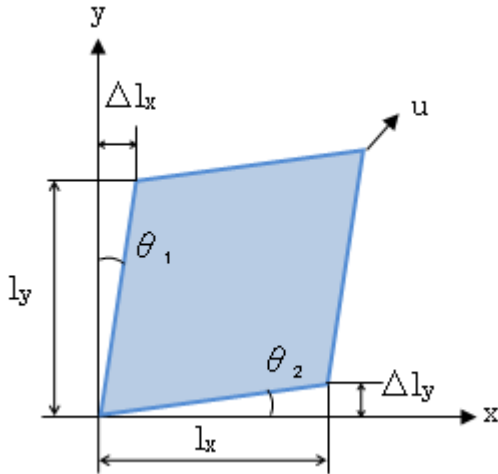
For “Masonry Wall” and “Passive Damper”, the shear deformation is defined as follows:

### 1) Shear deformation in one direction



Shear strain is  $\tau = \Delta l / 1 \approx \theta$

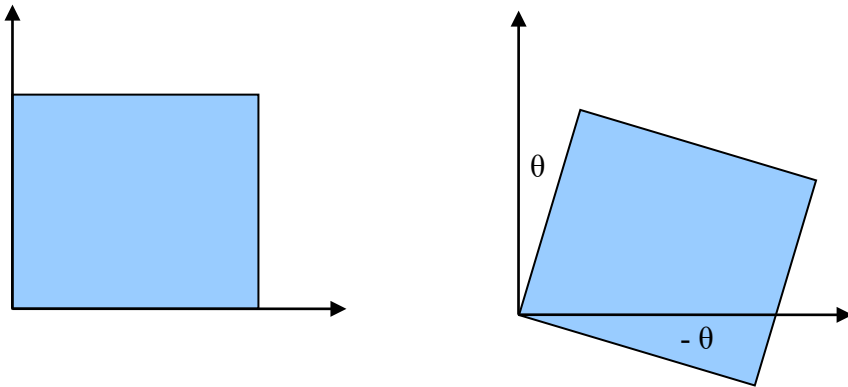
### 2) Shear deformation in two directions



Shear strain is  $\tau = \theta_1 + \theta_2 = \Delta l_x / l_y + \Delta l_y / l_x$

If we discuss small element  $\tau = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \rightarrow$  Eq. (2-7-4) and Eq. (2-8-2)

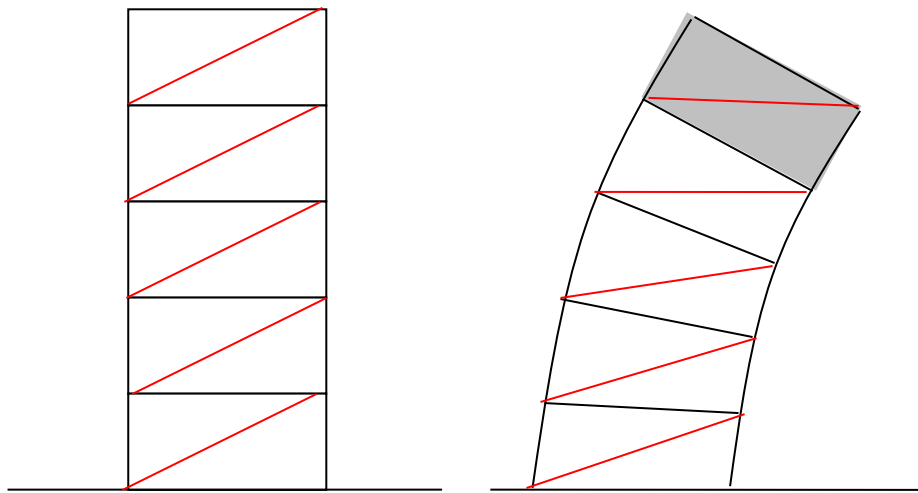
This definition is necessary to remove rotational component. To explain this, suppose there is only rotational (or bending) deformation,



From the above definition, shear angle will be

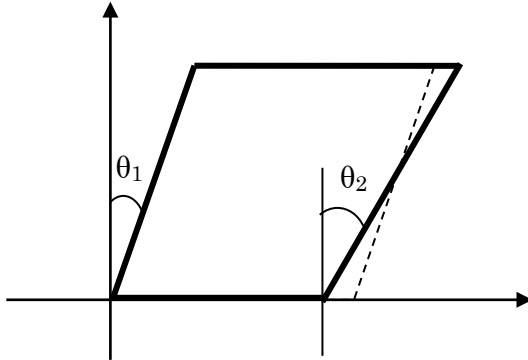
$$\tau = \theta + (-\theta) = 0$$

For example, in the upper story of the building under horizontal deformation, the bending component is dominant and the shear component is small. Therefore, the brace damper doesn't work in the upper story.



### 3) In case of damper element

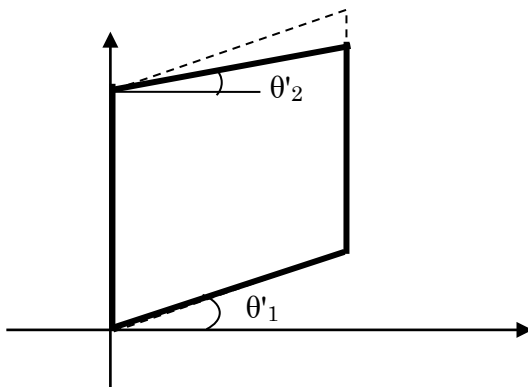
We define the shear angle in one direction as follows:



We adopt the average angle,

$$\theta = 1/2 (\theta_1 + \theta_2)$$

In the same way, the shear angle in another direction is



$$\theta' = 1/2 (\theta'_1 + \theta'_2)$$

## 2.9 Floor Element

In the default setting, STERA 3D adopts “rigid floor”. However, elastic deformation of a floor diaphragm in-plane can be considered by the option menu selecting “flexible floor”. The stiffness matrix of the floor element is constructed using a two dimensional isoparametric element.

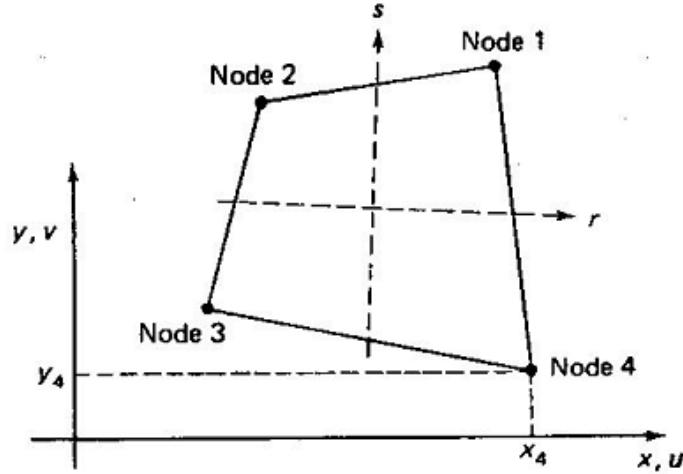


Figure 2-9-1 4-nodes isoparametric element

The stiffness matrix with 4-nodes isoparametric is expressed as,

$$\begin{Bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \\ P_4 \\ Q_4 \end{Bmatrix} = [K_F] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\mathbf{F} = \mathbf{K} \mathbf{u} \quad (2-9-1)$$

The coordinate transfer function  $\{x, y\}$  is expressed using the interpolation functions as follows:

$$\begin{aligned} x(r, s) &= \sum_{i=1}^4 h_i(r, s) x_i = \frac{1}{4}(1+r)(1+s)x_1 + \frac{1}{4}(1-r)(1+s)x_2 + \frac{1}{4}(1-r)(1-s)x_3 + \frac{1}{4}(1+r)(1-s)x_4 \\ y(r, s) &= \sum_{i=1}^4 h_i(r, s) y_i = \frac{1}{4}(1+r)(1+s)y_1 + \frac{1}{4}(1-r)(1+s)y_2 + \frac{1}{4}(1-r)(1-s)y_3 + \frac{1}{4}(1+r)(1-s)y_4 \end{aligned} \quad (2-9-2)$$

The deformation function  $\{u, v\}$  is also expressed using the same interpolation functions.

$$u(r, s) = \sum_{i=1}^4 h_i(r, s) u_i = \frac{1}{4}(1+r)(1+s)u_1 + \frac{1}{4}(1-r)(1+s)u_2 + \frac{1}{4}(1-r)(1-s)u_3 + \frac{1}{4}(1+r)(1-s)u_4$$

$$v(r, s) = \sum_{i=1}^4 h_i(r, s) v_i = \frac{1}{4}(1+r)(1+s)v_1 + \frac{1}{4}(1-r)(1+s)v_2 + \frac{1}{4}(1-r)(1-s)v_3 + \frac{1}{4}(1+r)(1-s)v_4$$
(2-9-3)

Stiffness matrix can be obtained from the “*Principle of Virtual Work Method*,” which is expressed in the following form:

$$\int_V \bar{\varepsilon}^T \sigma \, dv = \bar{u}^T F \quad (2-9-4)$$

where,  $\bar{\varepsilon}$  is a virtual strain vector,  $\sigma$  is a stress vector,  $\bar{u}$  is a virtual displacement vector and  $F$  is a load vector, respectively.

In case of the plane problem, the strain  $\varepsilon$  vector is defined as,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \quad (2-9-5)$$

Substituting equation (2-9-3) into equation (2-9-5), the strain vector is calculated from the nodal displacement vector as,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^4 \frac{\partial h_i}{\partial x} u_i \\ \sum_{i=1}^4 \frac{\partial h_i}{\partial y} v_i \\ \sum_{i=1}^4 \frac{\partial h_i}{\partial y} u_i + \sum_{i=1}^4 \frac{\partial h_i}{\partial x} v_i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 & \frac{\partial h_3}{\partial x} & 0 & \frac{\partial h_4}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} & 0 & \frac{\partial h_3}{\partial y} & 0 & \frac{\partial h_4}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial y} & \frac{\partial h_4}{\partial x} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}$$

$$\varepsilon = \mathbf{B} \mathbf{u} \quad (2-9-6)$$

In the plane stress problem, the stress-strain relationship is expressed as,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \quad (2-9-7)$$

$$\sigma = C \varepsilon$$

Substituting equation (2-9-6) into equation (2-9-7),

$$\sigma = C B u \quad (2-9-8)$$

From the Principle of Virtual Work Method,

$$\int_V (B \bar{u})^T (C B u) dv = \bar{u}^T \left( \int_{V(x,y)} B^T C B dx dy \right) u = \bar{u}^T F \quad (2-9-9)$$

Therefore, the stiffness equation is obtained as,

$$F = K u, \quad K = \int_V B^T C B dv \quad (2-9-10)$$

If we assume the constant thickness of the plate (= t), using the relation  $dv = t dx dy$ ,

$$K = t \int_{V(x,y)} B^T C B dx dy \quad (2-9-11)$$

Since this integration is defined in x-y coordinate, we must transfer the coordinate into r-s coordinate to use the numerical integration method. Introducing the **Jacobian matrix**,

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix}; \text{Jacobian Matrix} \quad (2-9-12)$$

the above integration is expressed in r-s coordinate as,

$$K = t \int_{-1}^1 \int_{-1}^1 B(x(r,s), y(r,s))^T C B(x(r,s), y(r,s)) \frac{\partial(x,y)}{\partial(r,s)} dr ds \quad (2-9-13)$$

where

$$\frac{\partial(x,y)}{\partial(r,s)} = \det J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} \quad (2-9-14)$$

### Evaluation of Jacobian Matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^4 \frac{\partial h_i}{\partial r} x_i & \sum_{i=1}^4 \frac{\partial h_i}{\partial r} y_i \\ \sum_{i=1}^4 \frac{\partial h_i}{\partial s} x_i & \sum_{i=1}^4 \frac{\partial h_i}{\partial s} y_i \end{pmatrix} \quad (2-9-15)$$

### Evaluation of the matrix B

$$B = \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 & \frac{\partial h_3}{\partial x} & 0 & \frac{\partial h_4}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} & 0 & \frac{\partial h_3}{\partial y} & 0 & \frac{\partial h_4}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial y} & \frac{\partial h_4}{\partial x} \end{pmatrix} \quad (2-9-16)$$

The derivatives  $\frac{\partial h_1}{\partial x}, \dots, \frac{\partial h_4}{\partial x}, \frac{\partial h_1}{\partial y}, \dots, \frac{\partial h_4}{\partial y}$  are calculated as,

$$\begin{aligned} \frac{\partial h_1}{\partial x} &= \frac{\partial h_1}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial h_1}{\partial s} \frac{\partial s}{\partial x}, \quad \dots, \quad \frac{\partial h_4}{\partial x} = \frac{\partial h_4}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial h_4}{\partial s} \frac{\partial s}{\partial x}, \\ \frac{\partial h_1}{\partial y} &= \frac{\partial h_1}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial h_1}{\partial s} \frac{\partial s}{\partial y}, \quad \dots, \quad \frac{\partial h_4}{\partial y} = \frac{\partial h_4}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial h_4}{\partial s} \frac{\partial s}{\partial y} \end{aligned}$$

In a matrix form,

$$\begin{aligned} \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial x} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_2}{\partial y} & \frac{\partial h_3}{\partial y} & \frac{\partial h_4}{\partial y} \end{pmatrix} &= \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_2}{\partial r} & \frac{\partial h_3}{\partial r} & \frac{\partial h_4}{\partial r} \\ \frac{\partial h_1}{\partial s} & \frac{\partial h_2}{\partial s} & \frac{\partial h_3}{\partial s} & \frac{\partial h_4}{\partial s} \end{pmatrix} \\ &= J^{-1} \begin{pmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_2}{\partial r} & \frac{\partial h_3}{\partial r} & \frac{\partial h_4}{\partial r} \\ \frac{\partial h_1}{\partial s} & \frac{\partial h_2}{\partial s} & \frac{\partial h_3}{\partial s} & \frac{\partial h_4}{\partial s} \end{pmatrix} \end{aligned} \quad (2-9-17)$$

### Evaluation of partial derivatives of the interpolation functions

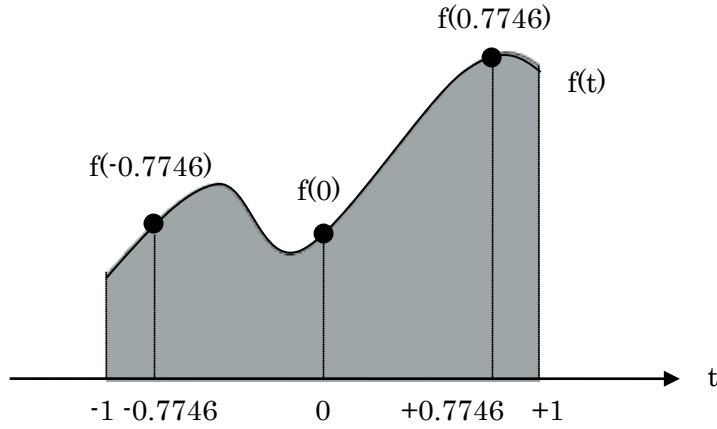
$$\begin{aligned} \frac{\partial h_1}{\partial r} &= \frac{1}{4}(1+s) & \frac{\partial h_1}{\partial s} &= \frac{1}{4}(1+r) \\ \frac{\partial h_2}{\partial r} &= -\frac{1}{4}(1+s) & \frac{\partial h_2}{\partial s} &= \frac{1}{4}(1-r) \\ \frac{\partial h_3}{\partial r} &= -\frac{1}{4}(1-s) & \frac{\partial h_3}{\partial s} &= -\frac{1}{4}(1-r) \\ \frac{\partial h_4}{\partial r} &= \frac{1}{4}(1-s) & \frac{\partial h_4}{\partial s} &= -\frac{1}{4}(1+s) \end{aligned}, \quad (2-9-18)$$

The **3 points Gauss Integration Formula** is defined as:

$$\begin{aligned} \int_{-1}^1 f(t) dt &= 0.5556 f(-0.7746) + 0.8889 f(0) + 0.5556 f(0.7746) \\ &= \alpha_1 f(t_1) + \alpha_2 f(t_2) + \alpha_3 f(t_3) \end{aligned} \quad (2-9-19)$$

where,

$$\begin{aligned} \alpha_1 &= 0.5556, \quad \alpha_2 = 0.8889, \quad \alpha_3 = 0.5556 \\ t_1 &= -0.7746, \quad t_2 = 0, \quad t_3 = 0.7746 \end{aligned}$$



The stiffness matrix is then calculated numerically as follows:

$$\begin{aligned} K &= t \int_{-1}^1 \int_{-1}^1 B(x(r,s), y(r,s))^T CB(x(r,s), y(r,s)) \frac{\partial(x,y)}{\partial(r,s)} dr ds \\ &= t \int_{-1}^1 \int_{-1}^1 F(r,s) dr ds \\ &= t \sum_{i=1}^3 \sum_{j=1}^3 \alpha_i \alpha_j F(r_i, s_j) \end{aligned} \quad (2-9-20)$$

where

$$F(r,s) = B(x(r,s), y(r,s))^T CB(x(r,s), y(r,s)) \frac{\partial(x,y)}{\partial(r,s)}$$

$$\begin{aligned} \alpha_1 &= 0.5556, \quad \alpha_2 = 0.8889, \quad \alpha_3 = 0.5556 \\ r_1 = s_1 &= -0.7746, \quad r_2 = s_2 = 0, \quad r_3 = s_3 = 0.7746 \end{aligned}$$

### From global node displacement to element node displacement

Transformation from global node displacements to element node displacements is,

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [T_{iF}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-9-21)$$

The component of the transformation matrix,  $[T_{iF}]$ , is discussed in Chapter 4 (Freedom Vector).

## 2.10 Connection Panel

### 1) General case

In the default setting, STERA3D assumes the rigid connection zone between column and beam. You can consider shear deformation of the connection area (we call “connection panel”) by the “Connection member” menu.

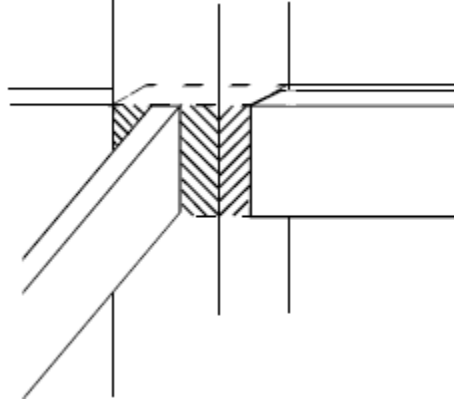


Figure 2-10-1 Connection area

Shear deformation of the connection panel,  $\gamma$ , is defined as shown in Figure 2-10-2.

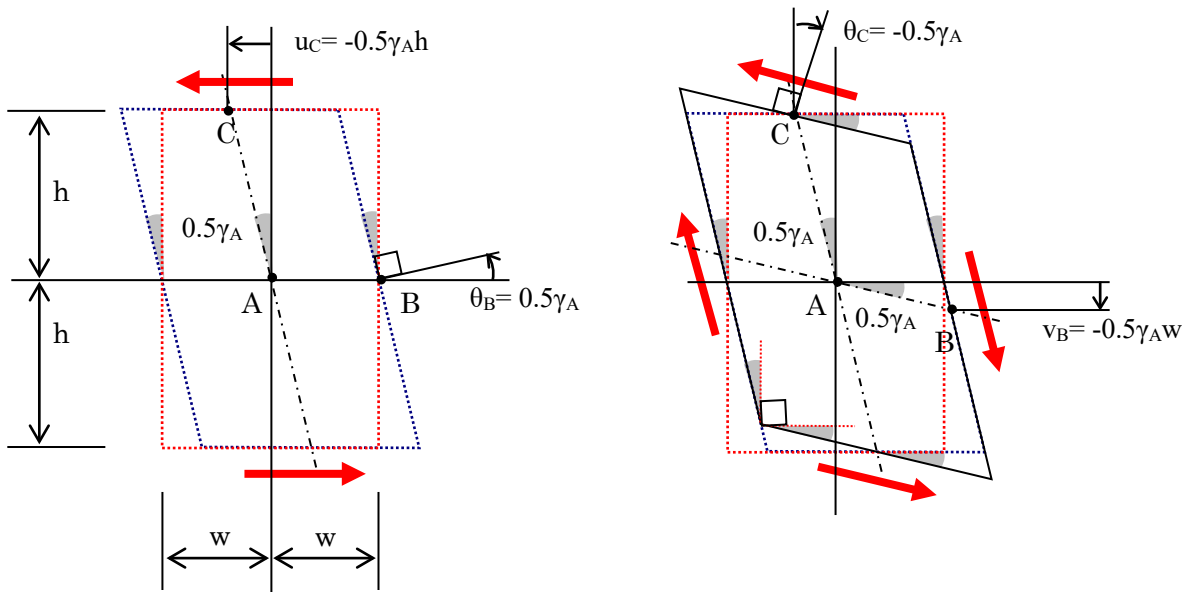


Figure 2-10-2 Definition of shear deformation

Differences of displacement at node B and C are;

$$\text{Node B: } \begin{Bmatrix} \Delta u_B \\ \Delta v_B \\ \Delta \theta_B \end{Bmatrix} \approx \begin{Bmatrix} 0 \\ -0.5\gamma_A w \\ 0.5\gamma_A \end{Bmatrix}, \quad \text{Node C: } \begin{Bmatrix} \Delta u_C \\ \Delta v_C \\ \Delta \theta_C \end{Bmatrix} \approx \begin{Bmatrix} -0.5\gamma_A h \\ 0 \\ -0.5\gamma_A \end{Bmatrix} \quad (2-10-1)$$

First we consider nodal movement without shear deformation of the connection panel. As shown in Figure 2-10-3, the displacement at node B and node C will be;

$$\text{Node B: } \begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix} \approx \begin{Bmatrix} u_A \\ v_A + \theta_A w \\ \theta_A \end{Bmatrix}, \quad \text{Node C: } \begin{Bmatrix} u_C \\ v_C \\ \theta_C \end{Bmatrix} \approx \begin{Bmatrix} u_A - \theta_A h \\ v_A \\ \theta_A \end{Bmatrix} \quad (2-10-2)$$

Then, we consider shear deformation of the connection as shown in Figure 2-10-4. By adding Equation (2-10-1) to (2-10-2), the displacement at node B and node C will be;

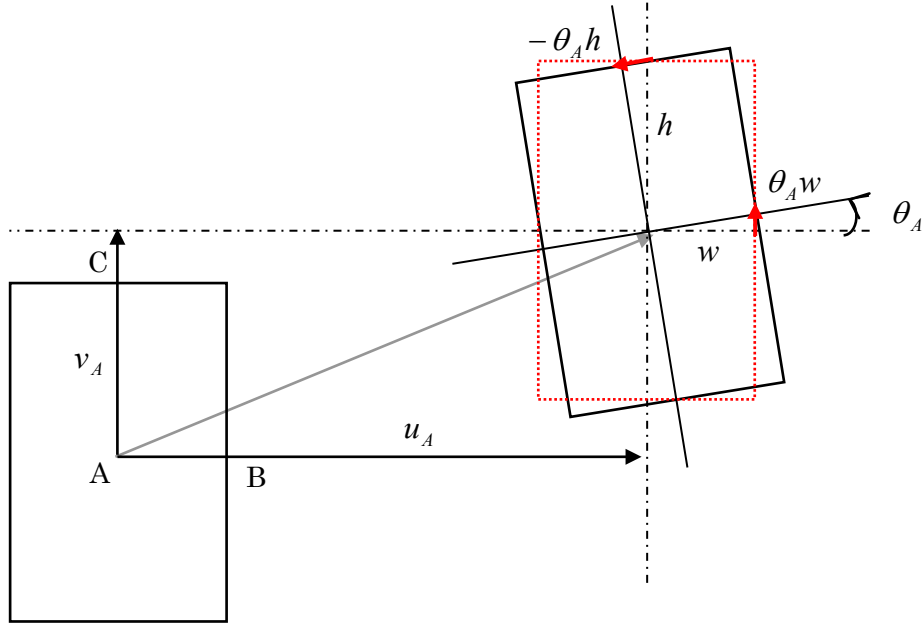


Figure 2-10-2 Nodal movement without shear deformation of the panel

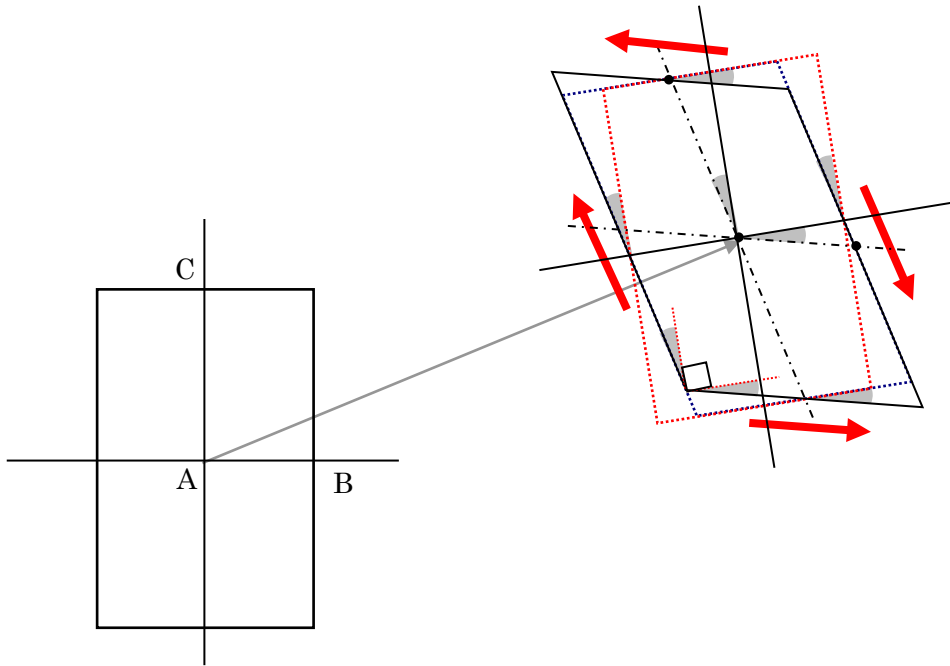


Figure 2-10-4 Nodal movement with shear deformation of the panel

Node B:

$$\begin{Bmatrix} u_B \\ v_B \\ \theta_B \end{Bmatrix} \approx \begin{Bmatrix} u_A \\ v_A + \theta_A w \\ \theta_A \end{Bmatrix} + \begin{Bmatrix} 0 \\ -0.5\gamma_A w \\ 0.5\gamma_A \end{Bmatrix} = \begin{Bmatrix} u_A \\ v_A + \theta_A w - 0.5\gamma_A w \\ \theta_A + 0.5\gamma_A \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & w & -0.5w \\ 0 & 0 & 1 & 0.5 \end{bmatrix} \begin{Bmatrix} u_A \\ v_A \\ \theta_A \\ \gamma_A \end{Bmatrix} \quad (2-10-3)$$

Node C:

$$\begin{Bmatrix} u_C \\ v_C \\ \theta_C \end{Bmatrix} \approx \begin{Bmatrix} u_A - \theta_A h \\ v_A \\ \theta_A \end{Bmatrix} + \begin{Bmatrix} -0.5\gamma_A h \\ 0 \\ -0.5\gamma_A \end{Bmatrix} = \begin{Bmatrix} u_A - \theta_A h - 0.5\gamma_A h \\ v_A \\ \theta_A - 0.5\gamma_A \end{Bmatrix} = \begin{bmatrix} 1 & 0 & h & -0.5h \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -0.5 \end{bmatrix} \begin{Bmatrix} u_A \\ v_A \\ \theta_A \\ \gamma_A \end{Bmatrix} \quad (2-10-4)$$

## 2) Beam element

In case of rigid connection, as described in Equation (2-1-7), the nodal displacement is expressed as,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \end{Bmatrix} = \begin{Bmatrix} \theta_{yA} - \tau \\ \theta_{yB} - \tau \end{Bmatrix}, \quad \tau = \frac{(u_{zB} - \lambda_B l' \theta_{yB}) - (u_{zA} + \lambda_A l' \theta_{yA})}{l'}$$

$$= \begin{Bmatrix} \theta_{yA} + \frac{1}{l'} u_{zA} + \lambda_A \theta_{yA} - \frac{1}{l'} u_{zB} + \lambda_B \theta_{yB} \\ \theta_{yB} + \frac{1}{l'} u_{zA} + \lambda_A \theta_{yA} - \frac{1}{l'} u_{zB} + \lambda_B \theta_{yB} \end{Bmatrix} = \begin{bmatrix} \frac{1}{l'} & -\frac{1}{l'} & 1 + \lambda_A & \lambda_B \\ \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1 + \lambda_B \end{bmatrix} \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \end{Bmatrix} \quad (2-10-5)$$

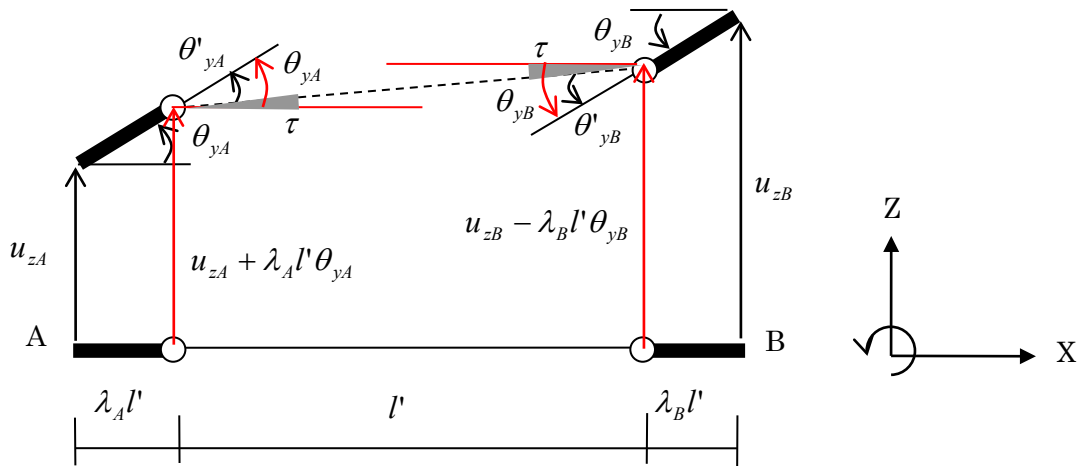


Figure 2-10-5 Beam displacement with rigid connection

If we consider shear deformation of connection panel, from Figure 2-10-6,

$$\begin{aligned}
 \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \end{Bmatrix} &= \begin{Bmatrix} \theta_{yA} + 0.5\gamma_{yA} - \tau \\ \theta_{yB} + 0.5\gamma_{yB} - \tau \end{Bmatrix}, \quad \tau = \frac{(u_{zB} - \lambda_B l'(\theta_{yB} - 0.5\gamma_{yB})) - (u_{zA} + \lambda_A l'(\theta_{yA} - 0.5\gamma_{yA}))}{l'} \\
 &= \begin{Bmatrix} \theta_{yA} + \frac{1}{l'} u_{zA} + \lambda_A \theta_{yA} - \frac{1}{l'} u_{zB} + \lambda_B \theta_{yB} + 0.5\gamma_{yA} - 0.5\lambda_A \gamma_{yA} - 0.5\lambda_B \gamma_{yB} \\ \theta_{yB} + \frac{1}{l'} u_{zA} + \lambda_A \theta_{yA} - \frac{1}{l'} u_{zB} + \lambda_B \theta_{yB} + 0.5\gamma_{yB} - 0.5\lambda_A \gamma_{yA} - 0.5\lambda_B \gamma_{yB} \end{Bmatrix} \\
 &= \begin{bmatrix} \frac{1}{l'} & -\frac{1}{l'} & 1+\lambda_A & \lambda_B & 0.5-0.5\lambda_A & -0.5\lambda_B \\ \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1+\lambda_B & -0.5\lambda_A & 0.5-0.5\lambda_B \end{bmatrix} \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \gamma_{yA} \\ \gamma_{yB} \end{Bmatrix} \quad (2-10-6)
 \end{aligned}$$

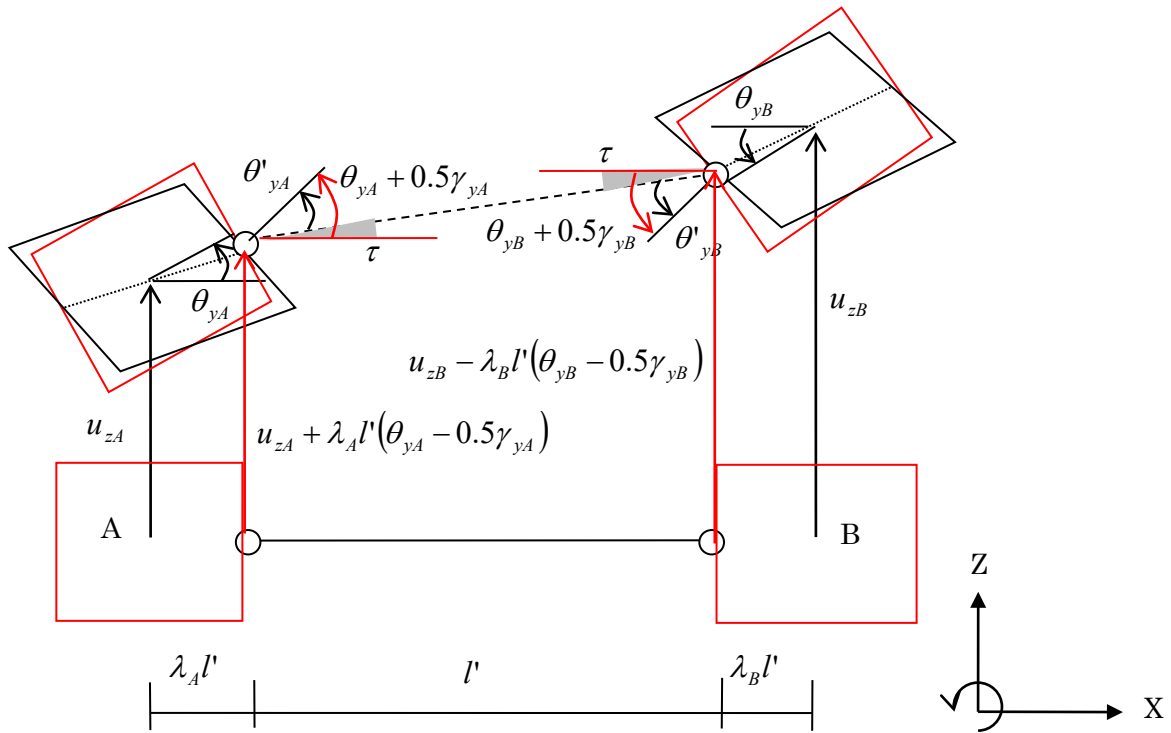


Figure 2-10-6 Beam displacement with shear deformation of connection panel

The transformation matrices for beam element are;

**Including connection panel and node movement**

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = \begin{bmatrix} \frac{1}{l'} & -\frac{1}{l'} & 1+\lambda_A & \lambda_B & 0.5-0.5\lambda_A & -0.5\lambda_B \\ \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1+\lambda_B & -0.5\lambda_A & 0.5-0.5\lambda_B \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \gamma_{yA} \\ \gamma_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = [\Lambda_B] \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \gamma_{yA} \\ \gamma_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} \quad (2-10-10)$$

**From global node displacement to element node displacement**

$$\begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \gamma_{yA} \\ \gamma_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = [T_{ixB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-10-11)$$

**From global node displacement to element face displacement**

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta'_{x} \end{Bmatrix} = [n_B] [\Lambda_B] [T_{ixB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{xB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-10-12)$$

**In case of Y-direction beam**

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{Y-Beam} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}_{Global} \quad (2-10-13)$$

$$\begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \gamma_{yA} \\ \gamma_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix}_{Y-Beam} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{xA} \\ \theta_{xB} \\ \gamma_{xA} \\ \gamma_{xB} \\ \delta_{yA} \\ \delta_{yB} \end{Bmatrix}_{Global} = [s_B] \begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{xA} \\ \theta_{xB} \\ \gamma_{xA} \\ \gamma_{xB} \\ \delta_{yA} \\ \delta_{yB} \end{Bmatrix}_{Global} \quad (2-10-14)$$

Transformation from the global node displacement to the element node displacement is,

$$\begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{xA} \\ \theta_{xB} \\ \gamma_{xA} \\ \gamma_{xB} \\ \delta_{yA} \\ \delta_{yB} \end{Bmatrix} = [T_{iyB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-10-15)$$

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \delta'_x \end{Bmatrix} = [n_B] [\Lambda_B] [s_B] [T_{iyB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_{yB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-10-16)$$

### 3) Column element

In case of rigid connection, as described in Equation (2-2-16), the nodal displacement in X-Z plane is expressed as,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \end{Bmatrix} = \begin{Bmatrix} \theta_{yA} - \tau \\ \theta_{yB} - \tau \end{Bmatrix}, \quad \tau = \frac{(u_{xA} - \lambda_A l' \theta_{yA}) - (u_{xB} + \lambda_B l' \theta_{yB})}{l'}$$

$$= \begin{Bmatrix} \theta_{yA} - \frac{1}{l'} u_{xA} + \lambda_A \theta_{yA} + \frac{1}{l'} u_{xB} + \lambda_B \theta_{yB} \\ \theta_{yB} - \frac{1}{l'} u_{xA} + \lambda_A \theta_{yA} + \frac{1}{l'} u_{xB} + \lambda_B \theta_{yB} \end{Bmatrix} = \begin{bmatrix} -\frac{1}{l'} & \frac{1}{l'} & 1 + \lambda_A & \lambda_B \\ -\frac{1}{l'} & \frac{1}{l'} & \lambda_A & 1 + \lambda_B \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \end{Bmatrix} \quad (2-10-17)$$

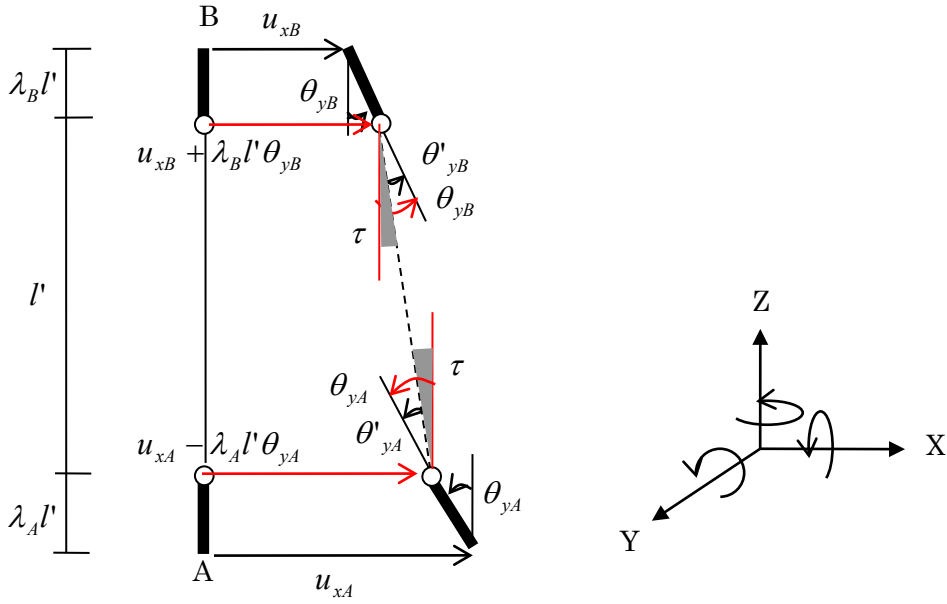


Figure 2-9-7 Column displacement with rigid connection (X-Z plane)

If we consider shear deformation of connection panel, from Figure 2-10-8,

$$\begin{aligned}
 \begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \end{Bmatrix} &= \begin{Bmatrix} \theta_{yA} - 0.5\gamma_{yA} - \tau \\ \theta_{yB} - 0.5\gamma_{yB} - \tau \end{Bmatrix}, \quad \tau = \frac{(u_{xA} - \lambda_A l'(\theta_{yA} + 0.5\gamma_{yA})) - (u_{xB} + \lambda_B l'(\theta_{yB} + 0.5\gamma_{yB}))}{l'} \\
 &= \begin{Bmatrix} \theta_{yA} - \frac{1}{l'} u_{xA} + \lambda_A \theta_{yA} + \frac{1}{l'} u_{xB} + \lambda_B \theta_{yB} - 0.5\gamma_{yA} + 0.5\lambda_A \gamma_{yA} + 0.5\lambda_B \gamma_{yB} \\ \theta_{yB} - \frac{1}{l'} u_{xA} + \lambda_A \theta_{yA} + \frac{1}{l'} u_{xB} + \lambda_B \theta_{yB} - 0.5\gamma_{yB} + 0.5\lambda_A \gamma_{yA} + 0.5\lambda_B \gamma_{yB} \end{Bmatrix} \\
 &= \begin{bmatrix} -\frac{1}{l'} & \frac{1}{l'} & 1 + \lambda_A & \lambda_B & -0.5 + 0.5\lambda_A & 0.5\lambda_B \\ -\frac{1}{l'} & \frac{1}{l'} & \lambda_A & 1 + \lambda_B & 0.5\lambda_A & -0.5 + 0.5\lambda_B \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \\ \gamma_{yA} \\ \gamma_{yB} \end{Bmatrix} \quad (2-10-18)
 \end{aligned}$$

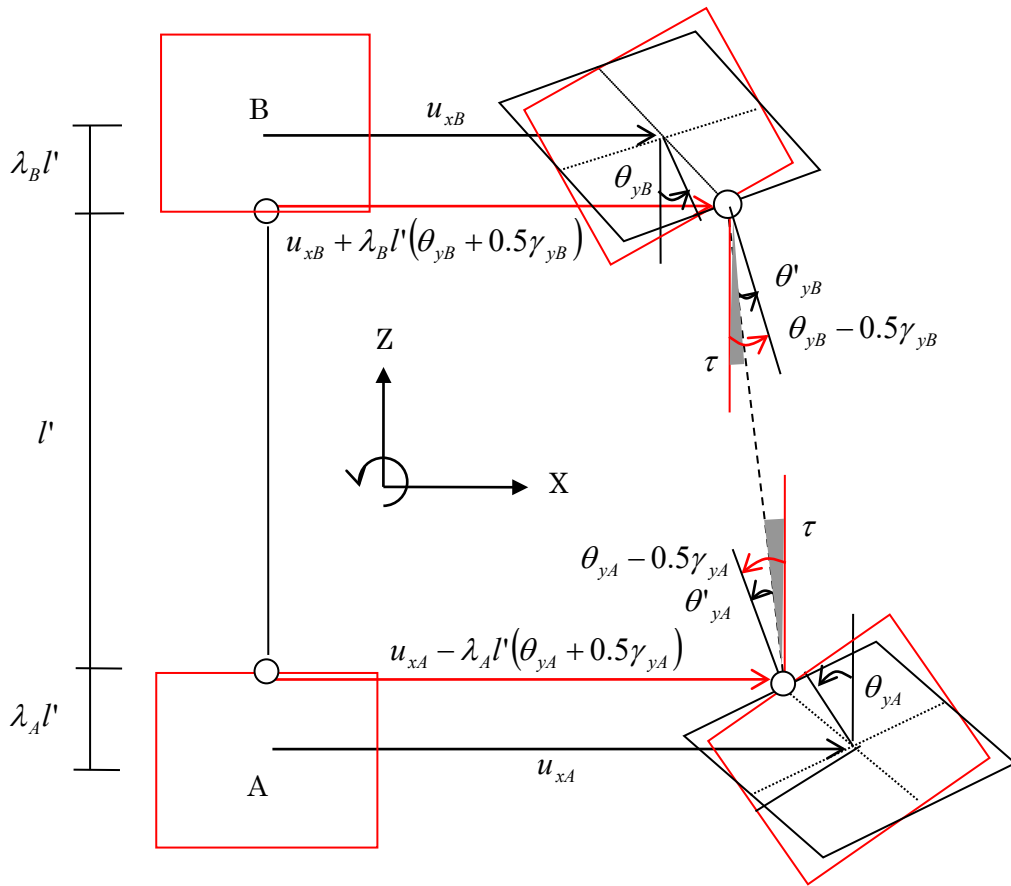


Figure 2-9-8 Column displacement with shear deformation of connection panel (X-Z plane)

In the same manner, assuming rigid connection, the nodal displacement of column in Y-Z plane is expressed as,

$$\begin{Bmatrix} \theta'_{xA} \\ \theta'_{xB} \end{Bmatrix} = \begin{Bmatrix} \theta_{xA} - \tau \\ \theta_{xB} - \tau \end{Bmatrix}, \quad \tau = \frac{(u_{yB} - \lambda_A l' \theta_{xB}) - (u_{yA} + \lambda_A l' \theta_{xA})}{l'}$$

$$= \begin{Bmatrix} \theta_{xA} + \frac{1}{l'} u_{yA} + \lambda_A \theta_{xA} - \frac{1}{l'} u_{yB} + \lambda_B \theta_{xB} \\ \theta_{xB} + \frac{1}{l'} u_{yA} + \lambda_A \theta_{xA} - \frac{1}{l'} u_{yB} + \lambda_B \theta_{xB} \end{Bmatrix} = \begin{bmatrix} \frac{1}{l'} & -\frac{1}{l'} & 1 + \lambda_A & \lambda_B \\ \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1 + \lambda_B \end{bmatrix} \begin{Bmatrix} u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \end{Bmatrix} \quad (2-10-19)$$

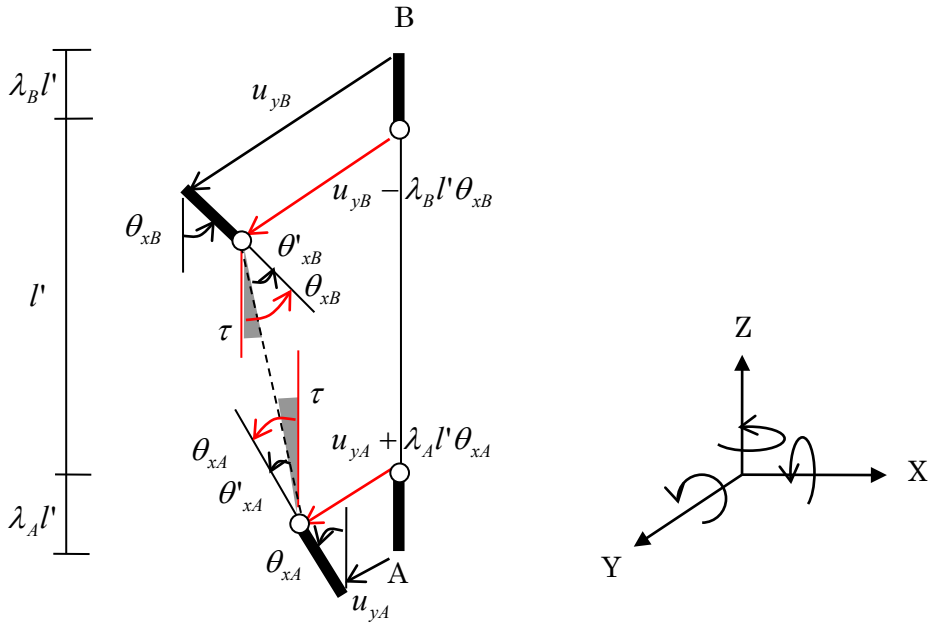


Figure 2-9-9 Column displacement with rigid connection (Y-Z plane)

If we consider shear deformation of connection panel, from Figure 2-10-10,

$$\begin{aligned}
 \begin{Bmatrix} \theta'_{xA} \\ \theta'_{xB} \end{Bmatrix} &= \begin{Bmatrix} \theta_{xA} - 0.5\gamma_{xA} - \tau \\ \theta_{xB} - 0.5\gamma_{xB} - \tau \end{Bmatrix}, \quad \tau = \frac{(u_{yB} - \lambda_B l'(\theta_{xB} + 0.5\gamma_{xB})) - (u_{yA} + \lambda_A l'(\theta_{xA} + 0.5\gamma_{xA}))}{l'} \\
 &= \begin{Bmatrix} \theta_{xA} + \frac{1}{l'} u_{yA} + \lambda_A \theta_{xA} - \frac{1}{l'} u_{yB} + \lambda_B \theta_{xB} - 0.5\gamma_{xA} + 0.5\lambda_A \gamma_{xA} + 0.5\lambda_B \gamma_{xB} \\ \theta_{xB} + \frac{1}{l'} u_{yA} + \lambda_A \theta_{xA} - \frac{1}{l'} u_{yB} + \lambda_B \theta_{xB} - 0.5\gamma_{xB} + 0.5\lambda_A \gamma_{xA} + 0.5\lambda_B \gamma_{xB} \end{Bmatrix} \\
 &= \begin{bmatrix} \frac{1}{l'} & -\frac{1}{l'} & 1+\lambda_A & \lambda_B & -0.5+0.5\lambda_A & 0.5\lambda_B \\ \frac{1}{l'} & -\frac{1}{l'} & \lambda_A & 1+\lambda_B & 0.5\lambda_A & -0.5+0.5\lambda_B \end{bmatrix} \begin{Bmatrix} u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \\ \gamma_{xA} \\ \gamma_{xB} \end{Bmatrix} \quad (2-10-20)
 \end{aligned}$$

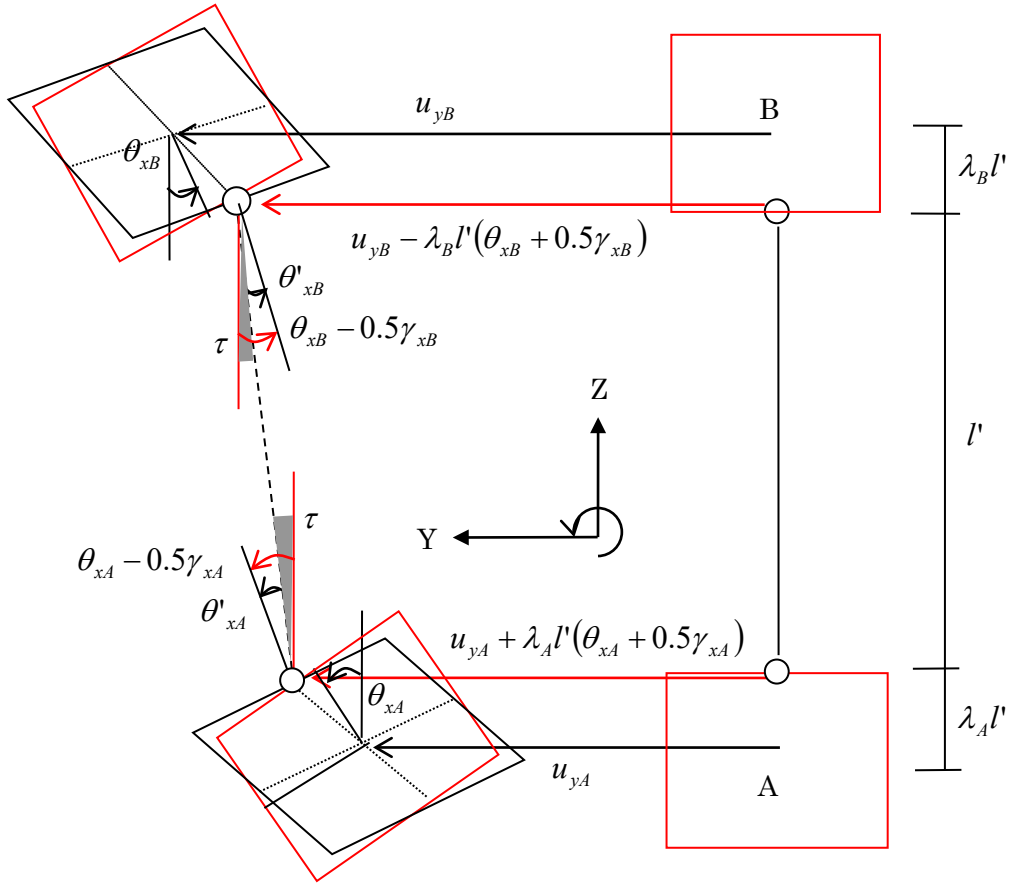


Figure 2-9-10 Column displacement with shear deformation of connection panel (Y-Z plane)

### Including connection panel and node movement

(2-10-21)

### From global node displacement to element node displacement

$$\begin{Bmatrix} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \\ \gamma_{yA} \\ \gamma_{yB} \\ u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \\ \gamma_{xA} \\ \gamma_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} = [T_{iC}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-10-22)$$

### From global node displacement to element face displacement

Transformation from the global node displacement to the element face displacement is,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta'_z \\ \theta'_z \end{Bmatrix} = [n_C] [\Lambda_C] [T_{iC}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [T_C] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-10-23)$$

#### 4) Force-displacement relationship for the connection

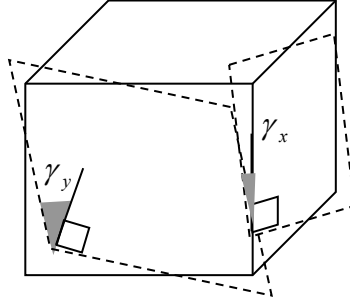


Figure 2-9-11 Shear deformation of connection area

The relationship between the displacement vector and force vector of the element is expressed as follows:

$$\begin{Bmatrix} M_{Px} \\ M_{Py} \end{Bmatrix} = \begin{bmatrix} k_{Px} & 0 \\ 0 & k_{Py} \end{bmatrix} \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} \quad (2-10-24)$$

where, initial stiffness of connection area is,

$$k_{Px} = k_{Py} = GV \quad (2-10-25)$$

where,  $G$  is the shear modulus and  $V$  is the volume of the connection.

#### From global node displacement to element node displacement

Transformation from the global node displacement to the element node displacement is,

$$\begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} = [T_P] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-10-26)$$

The component of the transformation matrix,  $[T_P]$ , is discussed in Chapter 4 (Freedom Vector).

#### Constitutive equation

The constitutive equation of the external spring is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_P] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-10-27)$$

where,

$$[K_P] = [T_P]^T [k_P] [T_P] \quad (2-10-28)$$

## 2.11 Ground Spring

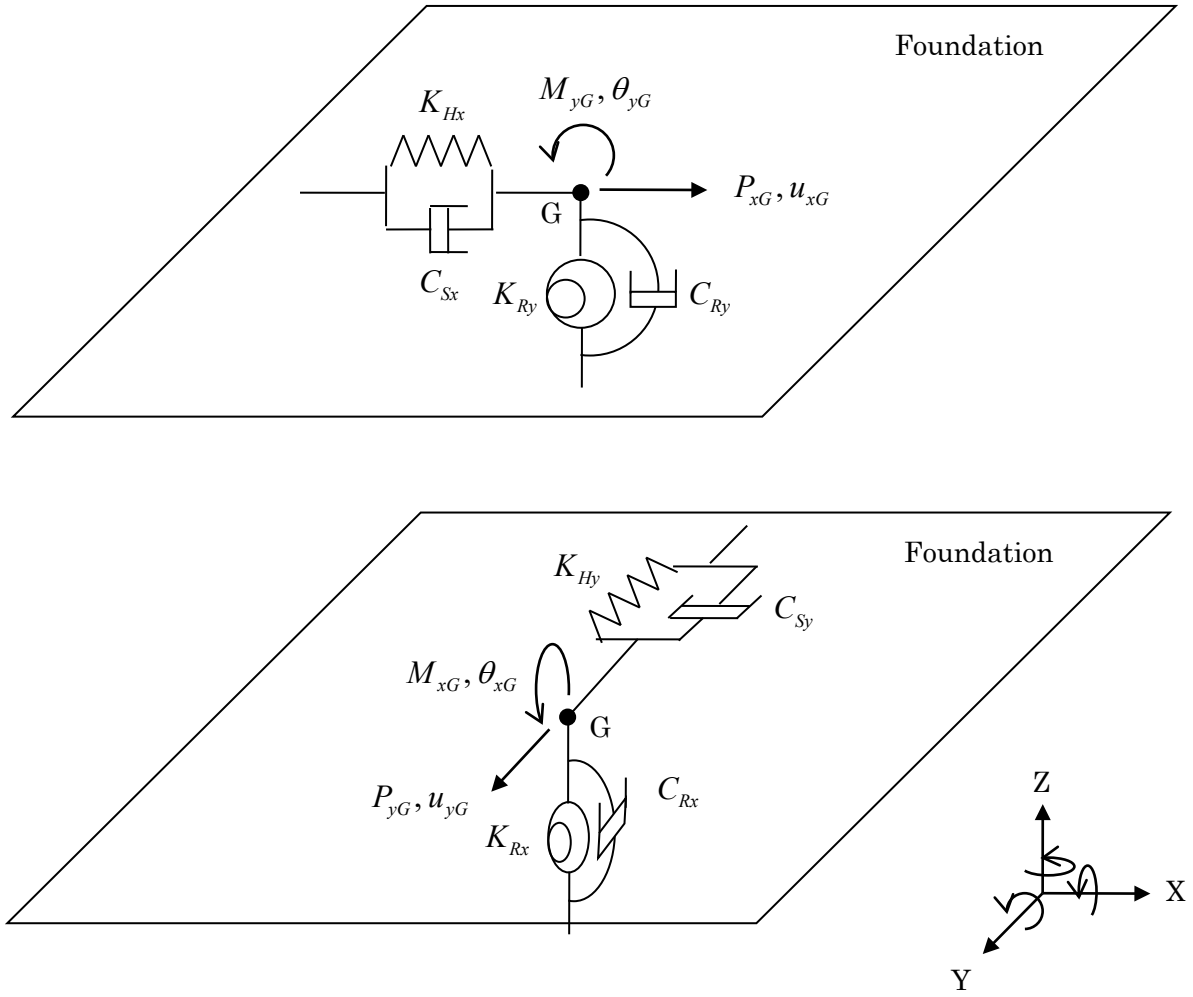


Figure 2-11-1 Element model for ground spring

### Force-displacement relationship for the element

The relationship between the displacement vector and force vector of the ground springs attached at the center of gravity of the foundation in Figure 2-11-1 is expressed as follows:

Sway and rocking in X-direction

$$\begin{Bmatrix} P_{xG} \\ M_{yG} \end{Bmatrix} = \begin{bmatrix} K_{Hx} & 0 \\ 0 & K_{Ry} \end{bmatrix} \begin{Bmatrix} u_{xG} \\ \theta_{yG} \end{Bmatrix} + \begin{bmatrix} C_{Hx} & 0 \\ 0 & C_{Ry} \end{bmatrix} \begin{Bmatrix} \dot{u}_{xG} \\ \dot{\theta}_{yG} \end{Bmatrix} \quad (2-11-1)$$

Sway and rocking in Y-direction

$$\begin{Bmatrix} P_{yG} \\ M_{xG} \end{Bmatrix} = \begin{bmatrix} K_{Hy} & 0 \\ 0 & K_{Rx} \end{bmatrix} \begin{Bmatrix} u_{yG} \\ \theta_{xG} \end{Bmatrix} + \begin{bmatrix} C_{Hy} & 0 \\ 0 & C_{Rx} \end{bmatrix} \begin{Bmatrix} \dot{u}_{yG} \\ \dot{\theta}_{xG} \end{Bmatrix} \quad (2-11-2)$$

Therefore

$$\begin{aligned}
 \begin{Bmatrix} P_{xG} \\ P_{yG} \\ M_{yG} \\ M_{xG} \end{Bmatrix} &= \begin{bmatrix} K_{Hx} & & & 0 \\ & K_{Hy} & & \\ & & K_{Ry} & \\ 0 & & & K_{Rx} \end{bmatrix} \begin{Bmatrix} u_{xG} \\ u_{yG} \\ \theta_{yG} \\ \theta_{xG} \end{Bmatrix} + \begin{bmatrix} C_{Hx} & & & 0 \\ & C_{Hy} & & \\ & & C_{Ry} & \\ 0 & & & C_{Rx} \end{bmatrix} \begin{Bmatrix} \dot{u}_{xG} \\ \dot{u}_{yG} \\ \dot{\theta}_{yG} \\ \dot{\theta}_{xG} \end{Bmatrix} \\
 &= [k_G] \begin{Bmatrix} u_{xG} \\ u_{yG} \\ \theta_{yG} \\ \theta_{xG} \end{Bmatrix} + [c_G] \begin{Bmatrix} \dot{u}_{xG} \\ \dot{u}_{yG} \\ \dot{\theta}_{yG} \\ \dot{\theta}_{xG} \end{Bmatrix}
 \end{aligned} \tag{2-11-3}$$

**From global node displacement to element node displacement**

$$\begin{Bmatrix} u_{xG} \\ u_{yG} \\ \theta_{yG} \\ \theta_{xG} \end{Bmatrix} = [T_G] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \tag{2-11-4}$$

The component of the transformation matrix,  $[T_G]$ , is discussed in Chapter 4 (Freedom Vector).

### Constitutive equation

The constitutive equation of the ground spring is;

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_G] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} + [C_G] \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_n \end{Bmatrix} \tag{2-11-5}$$

where,

$$[K_G] = [T_G]^T [k_G] [T_G], \quad [C_G] = [T_G]^T [c_G] [T_G] \tag{2-11-6}$$

### 3. Nonlinear Element Models

#### Notation

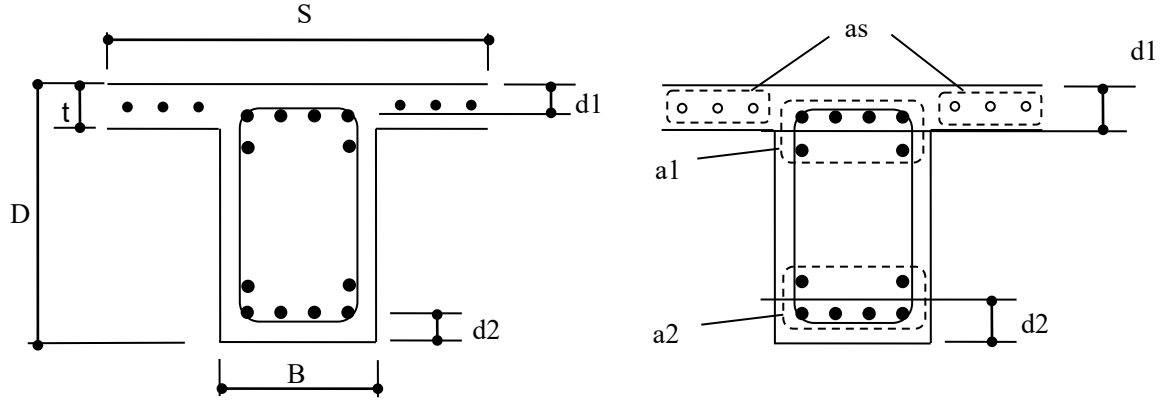
$a_t$	:	Area of rebar in the tension side of the section
$A_s$	:	Total area of rebar in the section
$\sigma_y$	:	Strength of rebar
$\sigma_B$	:	Compression strength of concrete
$\sigma_{wy}$	:	Strength of shear reinforcement
$D$	:	Depth of the section
$d$	:	Effective depth of the section.
$b$	:	Width of the beam
$j$	:	Distance between the centers of stress in the section $(= (7/8)d)$ .
$Z_e$	:	Section modulus including the slab effect.
$E_s$	:	Young's modulus of steel
$E_c$	:	Young's modulus of concrete
The Young's modulus of concrete $E_c$ (MPa) is calculated from the value of concrete strength $\sigma_B$ (MPa) by the following formula:		
$E_c = 3.35 \times 10^4 \times (\rho/24)^2 \times (\sigma_B/60)^{1/3}$		
where $\rho$ is the unit volume weight of concrete = 23 (kN/m <sup>3</sup> )		
$n$	:	Ratio of Young's modulus $(= E_s / E_c)$
$p_t$	:	Tensile reinforcement ratio
$p_w$	:	Shear reinforcement ratio
$I_e$	:	Moment of inertia of section considering the slab effect
$M_c$	:	Crack moment
$M_y$	:	Yield moment
$M/(QD)$	:	Shear span-to-depth ratio
$\theta_c$	:	Crack rotation of the beam end
$\theta_y$	:	Yield rotation of the beam end
$\phi_c$	:	Crack rotation of the nonlinear bending spring
$\phi_y$	:	Yield rotation of the nonlinear bending spring
$k_0$	:	Initial stiffness
$k_y$	:	Tangential stiffness at the yield point
$k_{y2}$	:	Stiffness after the yield point in the nonlinear bending spring

$k_{y3}$	:	Stiffness after the ultimate point in the nonlinear shear spring
$\alpha_y$	:	Stiffness degradation factor at the yield point
$Q_c$	:	Crack shear force
$Q_y$	:	Yield shear force
$Q_u$	:	Ultimate shear force
$x_s$	:	Distance between the corner springs in the Multi-spring model
$\gamma_c$	:	Crack shear deformation
$\gamma_y$	:	Yield shear deformation
$\gamma_u$	:	Ultimate shear deformation

### 3.1.1 Beam

#### 3.1.1 RC Beam

##### a) Section properties



- B : Width of beam,
- D : Height of beam,
- S : Effective width of slab,
- t : Thickness of slab
- d1 : Distance to the center of top main rebars,
- d2 : Distance to the center of bottom main rebars,
- a1 : Area of top main rebars,
- a2 : Area of bottom main rebars
- as : Area of rebars in slab

Figure 3-1-1 RC Beam Section

Area of section to calculate axial deformation

$$A_N = BD + (S - B)t + (n_E - 1)(a_1 + a_2 + a_s) \quad (3-1-1)$$

where,

$$n_E = E_s / E_c \quad : \text{Ratio of Young's modulus between steel } (E_s) \text{ and concrete } (E_c)$$

Area of section to calculate shear deformation

$$A_S = BD \quad (3-1-2)$$

Moment of inertia around the center of the section

$$I_e = \frac{BD^3}{12} + \frac{(S - B)t^3}{12} + BD \left( g - \frac{D}{2} \right)^2 + (S - B)t \left( D - \frac{t}{2} - g \right)^2 + (n_E - 1)a_1(D - d_1 - g)^2 + (n_E - 1)a_2(g - d_2)^2 + (n_E - 1)a_s \left( D - \frac{t}{2} - g \right)^2 \quad (3-1-3)$$

where, g is the center of beam section calculated by

$$g = \frac{BD^2 / 2 + (S - B)t(D - t / 2) + (n_E - 1)\{a_2 d_2 + a_1(D - d_1) + a_s(D - t / 2)\}}{A_N} \quad (3-1-4)$$

## b) Nonlinear bending spring

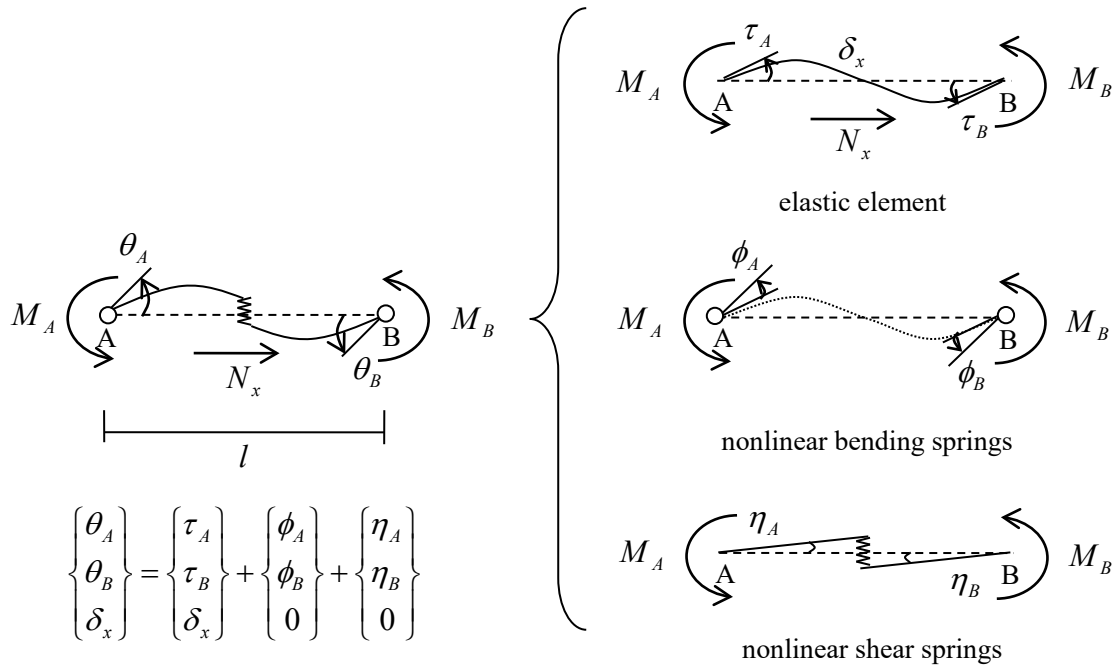


Figure 3-1-2 Element model for beam

Hysteresis model of a nonlinear bending spring is defined as the moment-rotation relationship under the anti-symmetry loading in Figure 3-1-3. The initial stiffness of the nonlinear spring is supposed to be infinite, however, in numerical calculation, a large enough value is used for the stiffness.

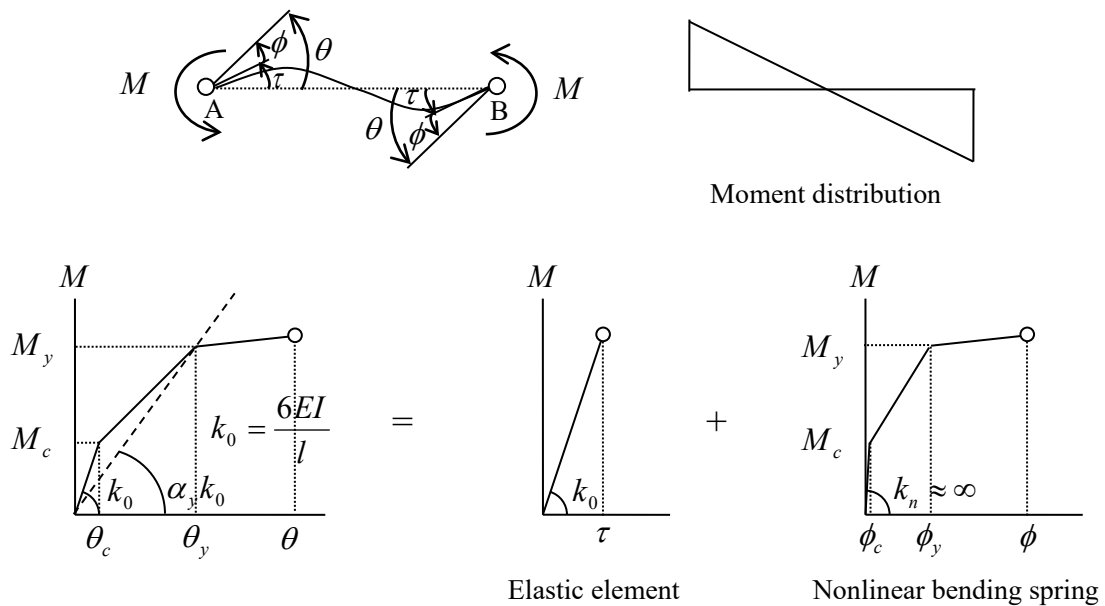


Figure 3-1-3 Moment – rotation relationship at bending spring

### Crack moment force

For reinforced concrete elements, the crack moment,  $M_c$  is calculated as,

$$M_{c1} = 0.56\sqrt{\sigma_B}Z_{e1}, \quad Z_{e1} = I_e / g \quad \text{when tension in top main rebars} \quad (3-1-5)$$

$$M_{c2} = 0.56\sqrt{\sigma_B}Z_{e2}, \quad Z_{e2} = I_e / (D - g) \quad \text{when tension in bottom main rebars} \quad (3-1-6)$$

where,

$\sigma_B$  : Compression strength of concrete (N/mm<sup>2</sup>)

$Z_{e1}, Z_{e2}$  : Section modulus

### Yield moment force

The yield moment,  $M_y$  is calculated as,

$$M_{y1} = 0.9a_1\sigma_y(D - d_1) + 0.9a_s\sigma_y(D - t/2) \quad \text{when tension in top main rebars} \quad (3-1-7)$$

$$M_{y2} = 0.9a_2\sigma_y(D - d_2) \quad \text{when tension in bottom main rebars} \quad (3-1-8)$$

where,

$\sigma_y$  : Strength of rebar (N/mm<sup>2</sup>)

### Yield rotation

The tangential stiffness at the yield point,  $k_y$ , is obtained from the following equation,:

$$k_y = \alpha_y k_0, \quad k_0 = \frac{6E_c I_e}{l} \quad (3-1-9)$$

where,

$\alpha_y$  is the stiffness degradation factor at the yield point, which is obtained from the following empirical formulas:

$$\alpha_y = (0.043 + 1.63np_t + 0.043a/D)(d/D)^2, \quad (a/D \leq 2) \quad (3-1-10)$$

$$\alpha_y = (-0.0836 + 0.159a/D)(d/D)^2, \quad (a/D > 2) \quad (3-1-11)$$

where,

$p_t$  : Tensile reinforcement ratio

$$p_t = (a_1 + a_s)/(BD) \quad \text{(when tension in top main rebars)}$$

$$p_t = (a_s)/(BD) \quad \text{(when tension in bottom main rebars)}$$

$a/D$  :  $\approx$  Shear span-to-depth ratio ( $= l/(2D)$ )

$d$  : Effective depth

$$d = D - d_1 \quad \text{(when tension in top main rebars)}$$

$$d = D - d_2 \quad \text{(when tension in bottom main rebars)}$$

$\alpha_y$  is modified in case of tension in top main rebars as

$$\alpha_y' = \alpha_y \frac{I_{e0}}{I_e} \quad (3-1-12)$$

where  $I_{e0} = \frac{BD^3}{12}$  : the moment of inertia of square section without slab

The yield rotation of the nonlinear bending beam,  $\phi_y$ , is then obtained from,

$$\phi_y = \left( \frac{1}{\alpha_y} - 1 \right) \frac{M_y}{k_0} \quad (3-1-13)$$

In general, the relation between the rotation of bending spring and that of nonlinear bending spring is

$$\phi = \theta - \frac{M_y}{k_0} \quad (3-1-14)$$

### Crack rotation

From Figure 3-1-2, the crack rotation of the nonlinear bending beam,  $\phi_c$ , is supposed to be zero value, however, in STERA\_3D program, it is assumed as,

$$\phi_c = 0.001 \phi_y \quad (3-1-15)$$

## Effective width of slab

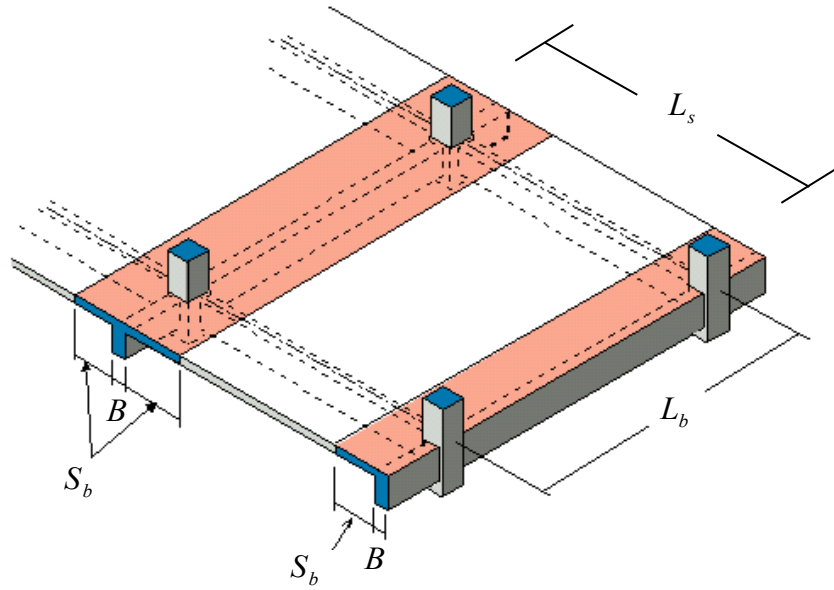


Figure 3-1-4 Effective slab area for flexural capacity of beam

In general, effective width of slab for the flexural behavior of a beam is assumed as,

$$S_b = 0.1 L_b \approx D \quad (3-1-16)$$

where,  $L_b$  : Length of beam  
 $D$  : Height of beam

However, recent studies suggest the contribution of full length of slab to the flexural capacity,  $M_y$ , of a beam. Therefore, STERA3D adopts two types of effective widths:

- 1) For calculating section area and moment of inertia

$$S_b = 0.1 L_b \approx D$$

- 2) For calculating the yield moment,  $M_y$ , in Equation (3-1-8),

$$S_b = \eta_s L_s \quad (3-1-17)$$

where,  $L_s$  : Length of span  
 $\eta_s$  : Effective slab ratio (0.1 ~ 0.5), the default value is 0.1.

### Hysteresis model

To consider the difference of the flexural capacity between positive and negative side of the beam, a degrading tri-linear slip model is developed based on the Takeda Model for the hysteresis model of the bending springs of the beam.

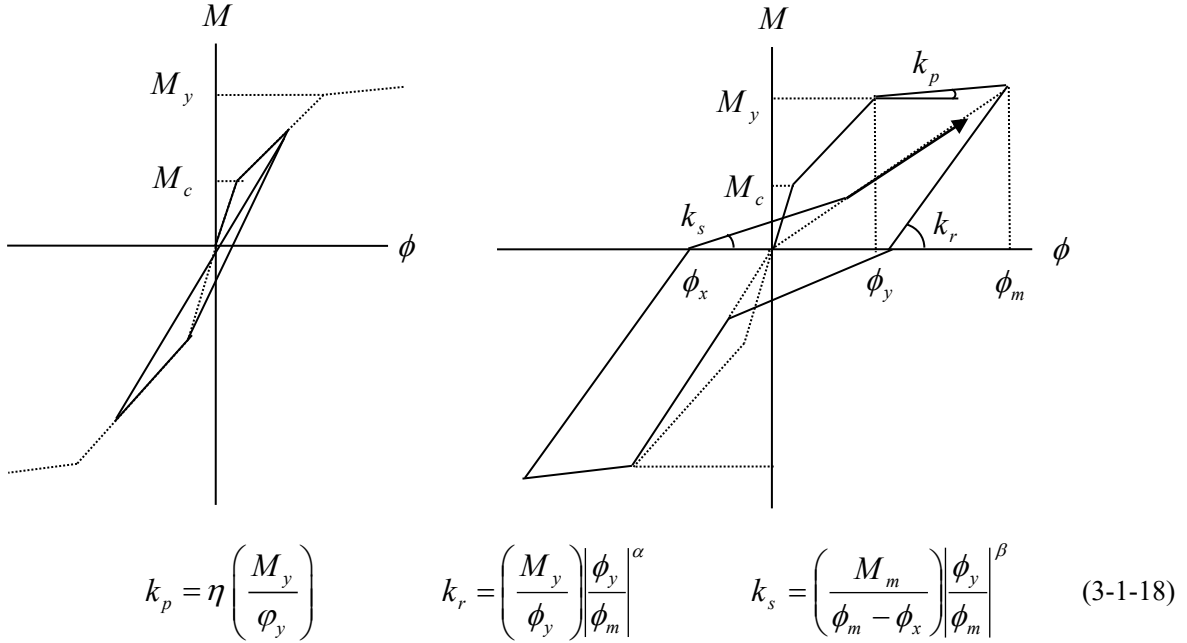


Figure 3-1-5 Degrading Tri-linear Slip Model  
( $\alpha=0.5$ ,  $\beta=0.0$  and  $\eta=0.001$  as default values)

The strength degradation under cyclic loading is considered by elongating the target displacement,  $\phi_m$ , to be  $\phi'_m$  as shown in the following Figure:

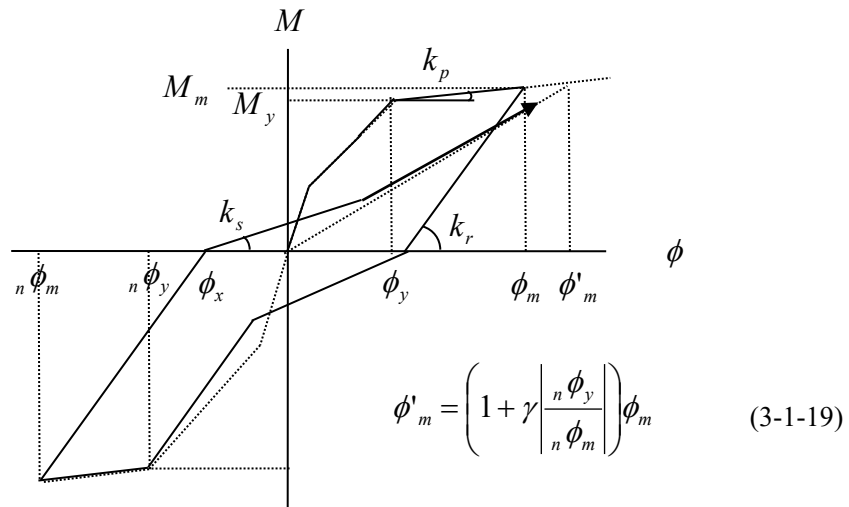


Figure 3-1-6 Introducing strength degradation ( $\gamma=0.0$  as default value)

### Relationship between curvature and rotation

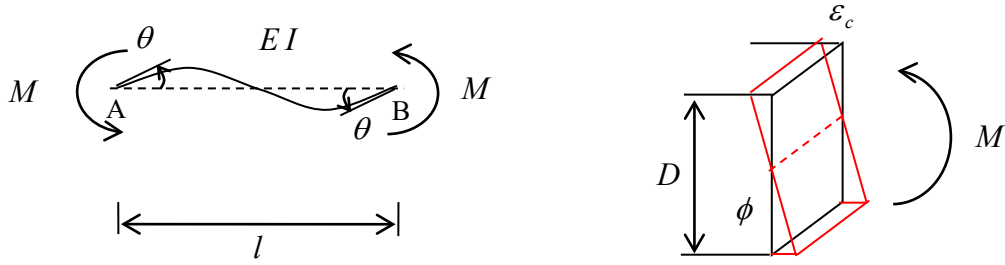


Figure 3-1-7 Rotation angle and curvature at beam ends

Let's think about the relationship between curvature and rotation at the end of a beam.

In the above loading condition, the relationship between moment and rotation is

$$M = \frac{6EI}{l} \theta \quad (3-1-20)$$

On the other hand, the relationship between moment and curvature is

$$\phi = \frac{M}{EI} \quad (3-1-21)$$

Therefore,

$$\phi = \frac{6}{l} \theta \quad (3-1-22)$$

Assuming the neutral axis is in the middle of the section, the relationship between curvature and compression strain at the section end is

$$\phi = \frac{\varepsilon_c}{D/2} \quad (3-1-23)$$

Therefore, the relationship between rotation and compressive strain is

$$\theta = \frac{l}{6} \phi = \frac{l}{3D} \varepsilon_c \quad (3-1-24)$$

Assuming  $D \approx \frac{l}{9}$ , then

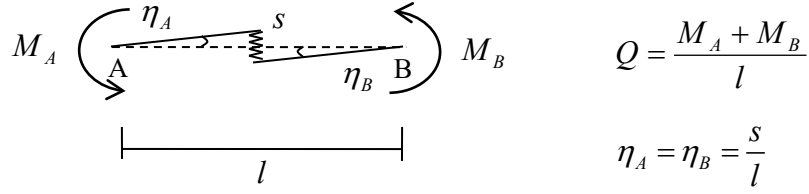
$$\theta = 3\varepsilon_c \quad (3-1-25)$$

If  $\varepsilon_c$  reaches 0.003,  $\theta$  is around 0.01 (=1/100).

It corresponds to the yielding rotation of a beam.

### c) Nonlinear shear spring

Hysteresis model of nonlinear shear spring is defined as the shear force – shear rotation relationship using an origin-oriented poly-linear model.



nonlinear shear springs

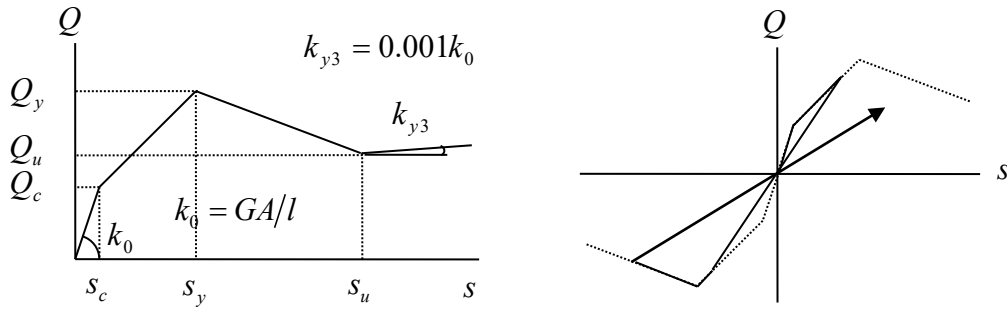


Figure 3-1-8 Force–deformation relationship of shear spring

### Yield shear force

The yield shear force,  $Q_y$  is calculated as,

$$Q_y = \left\{ \frac{0.053 p_t^{0.23} (\sigma_B + 18)}{M / (QD) + 0.12} + 0.85 \sqrt{p_w \cdot \sigma_{wy}} \right\} b \cdot j \quad (3-1-26)$$

where,

- $p_t$  : Tensile reinforcement ratio
- $\sigma_B$  : Compression strength of concrete
- $p_w$  : Shear reinforcement ratio
- $\sigma_{wy}$  : Strength of shear reinforcement
- $j$  : Distance between the centers of stress in the section  $(= (7/8)d)$ .

### Crack shear force

The crack shear force is,  $Q_c$ , is assumed as,

$$Q_c = \frac{Q_y}{3} \quad (3-1-27)$$

### Ultimate shear force

The ultimate shear force is,  $Q_u$ , is assumed as,

$$Q_u = Q_y + k_{y3} (s_u - s_y) \quad (3-1-28)$$

### NOTE)

In STERA\_3D, the stiffness after yielding is temporary assumed to be positive to avoid instability in numerical analysis.

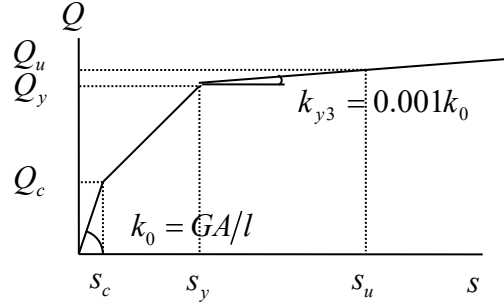


Figure 3-1-9 Stiffness after yielding

### Crack shear deformation

The crack shear deformation is obtained as,

$$s_c = \gamma_c l, \quad \gamma_c = \frac{Q_c}{GA} \quad (3-1-29)$$

### Yield shear displacement

The yield shear deformation is assumed as,

$$s_y = \gamma_y l, \quad \gamma_y = \frac{1}{250} \quad (3-1-30)$$

### Ultimate shear displacement

The ultimate shear deformation is assumed as,

$$s_u = \gamma_u l, \quad \gamma_u = \frac{1}{100} \quad (3-1-31)$$

#### d) Modification of initial stiffness of nonlinear springs

In numerical calculation, a large dummy value is used for the initial stiffness of the nonlinear spring to represent rigid condition. This large stiffness may cause an error for estimating the force from the displacement. One possible way to solve the problem is to reduce the initial stiffness of the nonlinear spring to be a certain value reasonable for calculation, and on the other hand, increase the stiffness of the elastic element so that the total initial stiffness of the beam element does not change from the original one. This idea is proposed by K-N Li (2004) for MS model.

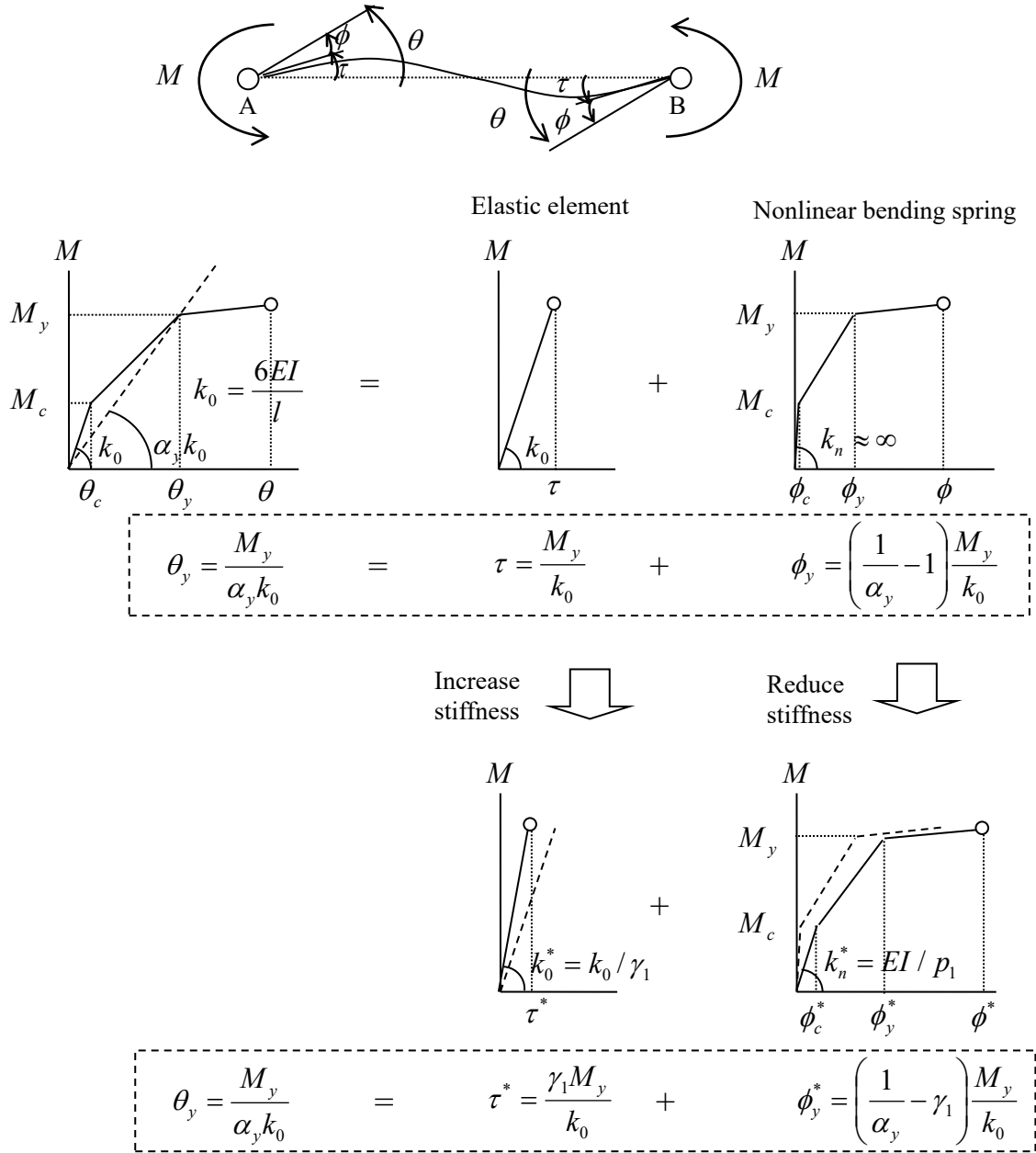


Figure 3-1-10 Modification of moment – rotation relationship

The idea is realized using flexibility reduction factors,  $\gamma_1 (< 1)$ ,  $\gamma_2 (< 1)$ , in the relationship between the displacement vector and force vector of the elastic element in Equation (2-1-1) as,

$$\begin{Bmatrix} \tau'_{yA} \\ \tau'_{yB} \\ \delta'_x \end{Bmatrix} = \begin{bmatrix} \gamma_1 \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ -\frac{l'}{6EI_y} & \gamma_2 \frac{l'}{3EI_y} & 0 \\ 0 & 0 & \frac{l'}{EA} \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ N'_x \end{Bmatrix} \quad (3-1-32)$$

It must be  $\gamma_1 \frac{l'}{3EI_y} > \frac{l'}{6EI_y}$  or  $\gamma_1 > 0.5$  and  $\gamma_2 \frac{l'}{3EI_y} > \frac{l'}{6EI_y}$  or  $\gamma_2 > 0.5$ .

Also the initial flexibility matrix of the nonlinear spring can be expressed as follows, introducing the parameters,  $p_1, p_2$  to increase the initial flexibility.

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{yB} \end{Bmatrix} = \begin{bmatrix} p_1/EI & 0 \\ 0 & p_2/EI \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \end{Bmatrix} \quad (3-1-33)$$

When  $p_1 \rightarrow 0, p_2 \rightarrow 0$ , it represents the infinite stiffness for rigid condition. Accordingly, the crack and yield rotation will be modified as,

$$\phi_c^* = p_1 \frac{M_c}{EI}, \quad \phi_y^* = \left( \frac{1}{\alpha_y} - \gamma_1 \right) \frac{M_y}{k_0} \quad (3-1-34)$$

In general, the relation between the rotation of bending spring and that of nonlinear bending spring is

$$\phi = \theta - \gamma_1 \frac{M_y}{k_0} \quad (3-1-35)$$

Making the modified flexibility matrix to be identical to the original one,

$$\begin{bmatrix} \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ & \frac{l'}{3EI_y} & 0 \\ sym. & & \frac{l'}{EA} \end{bmatrix}_{original} = \begin{bmatrix} \frac{p_1}{EI} + \gamma_1 \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ & \frac{p_2}{EI} + \gamma_2 \frac{l'}{3EI_y} & 0 \\ sym. & & \frac{l'}{EA} \end{bmatrix}_{modified} \quad (3-1-36)$$

This gives the flexibility reduction factors as:

$$\gamma_1 = 1 - \frac{3}{l'} p_1, \quad \gamma_2 = 1 - \frac{3}{l'} p_2 \quad (3-1-37)$$

From the conditions  $\gamma_1 > 0.5$  and  $\gamma_2 > 0.5$ ,

$$p_1 < \frac{l'}{6}, \quad p_2 < \frac{l'}{6} \quad (3-1-38)$$

K-N Li (2004) calls these parameters,  $p_1, p_2$ , as “plastic zones” and recommends to be  $p_1 = p_2 = \frac{l'}{10}$ .

Then the reduction factors will be  $\gamma_1 = \gamma_2 = 0.7$ .

**e) Modification of stiffness degradation factor at the yield point**

(The following modification of the stiffness degradation factor,  $\alpha_y$ , is suggested by Prof. Okano at Chiba University.)

From Equations (3-1-32) and (3-1-34), the yield rotation of the member  $\theta_y$  under anti-symmetric loading condition,  $M_A = M_B = M_y$ , is calculated as,

$$\theta_y = \frac{(2\gamma - 1)M_y}{k_0} + \left( \frac{1}{\alpha_y} - \gamma \right) \frac{M_y}{k_0} = \left( \frac{1}{\alpha_y} + \gamma - 1 \right) \frac{M_y}{k_0} \quad (3-1-39)$$

where  $\gamma_1 = \gamma_2 = \gamma$ .

The yield rotation  $\theta_y$  in Equation (3-1-39) is different from the formula in Figure 3-1-10 since the factor  $\gamma$  is multiplied to only diagonal elements of flexural matrix in Equation (3-1-32).

The stiffness degradation factor is then obtained as,

$$\frac{1}{\alpha'_y} = \left( \frac{1}{\alpha_y} + \gamma - 1 \right) \quad (3-1-40)$$

To realize the designated value of stiffness degradation factor,  $\alpha_y$  should be modified as,

$$\alpha_y = 1 / \left( \frac{1}{\alpha'_y} + 1 - \gamma \right) \quad (3-1-41)$$

For example, to realize the stiffness degradation factor  $\alpha'_y = 0.4$ , assuming  $\gamma = 0.7$ , the modified  $\alpha_y$  is

$$\alpha_y = 1 / \left( \frac{1}{0.4} + 1 - 0.7 \right) = 0.357$$

This modification is done automatically in STERA\_3D.

#### f) Modification of considering rigid zone ratio

A beam-column connection can be idealized as a rigid zone. In case of a beam element, the default length of the rigid zone is set to be a half of the column width, and the nonlinear bending spring of the beam element is arranged at the position of the column face.

On the other hand, if elastic deformation of the connection is considered by reducing the length of rigid zone, the position of the nonlinear bending spring will be inside the connection area. In this case, when the nonlinear bending spring is yielding, the moment value at the position of the column face is smaller than the yield moment.

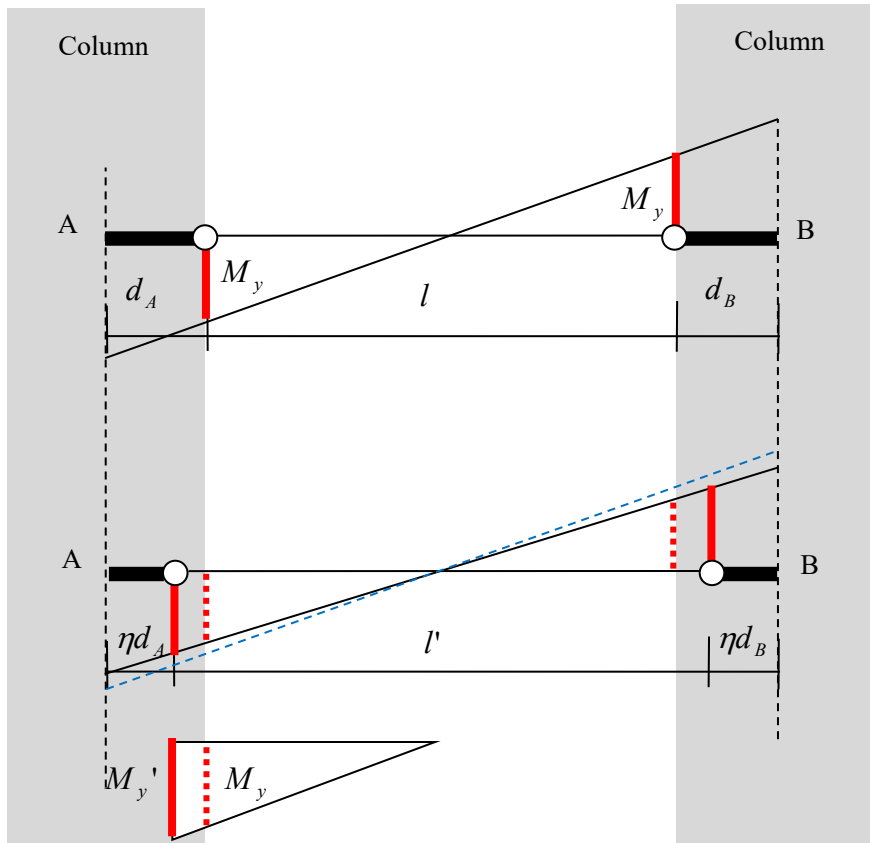


Figure 3-1-11 Reduction of rigid zone and modification of yield moment

To make the moment at the column face to be the same as yield moment, the yield moment of the nonlinear bending spring is increased as,

$$M_y' = \frac{l/2 + (1-\eta)d_A}{l/2} M_y = \xi M_y \quad (3-1-42)$$

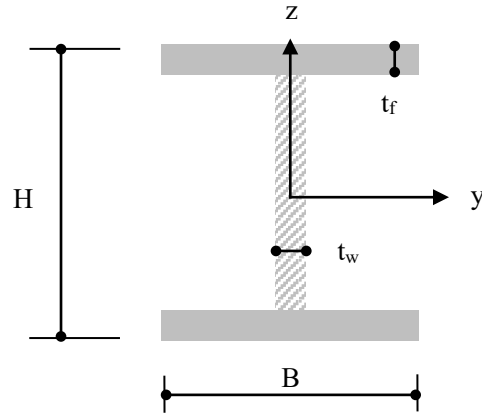
$$\xi = 1 + 2(1-\eta) \frac{d_A}{l}$$

For example, when  $l = 540\text{cm}$ ,  $d_A = 30\text{cm}$ ,  $\eta = 0.75$ ,

$$\xi = 1 + 2 \times (0.25) \times 30 / 540 = 1.027 \quad (3-1-43)$$

### 3.1.2 Steel Beam

#### a) Section properties




$B$  : Width,  $H$  : Height,  $t_w, t_f$  : Thickness

Figure 3-1-12 Steel Beam Section

Area of section to calculate axial deformation

$$A_N = 2Bt_f + (H - 2t_f)t_w \quad (3-1-44)$$

Area of section to calculate shear deformation (  )

$$A_S = (H - 2t_f)t_w \quad (3-1-45)$$

Moment of inertia around the center of the section

$$I_y = \frac{BH^3 - (B - t_w)(H - 2t_f)^3}{12} \quad : \text{along strong axis} \quad (3-1-46)$$

$$I_z = \frac{2t_f B^3 + (H - 2t_f)t_w^3}{12} \quad : \text{along weak axis} \quad (3-1-47)$$

Moment of inertia for torsion

$$I_x = I_y + I_z \quad (3-1-48)$$

**b) Nonlinear bending spring**

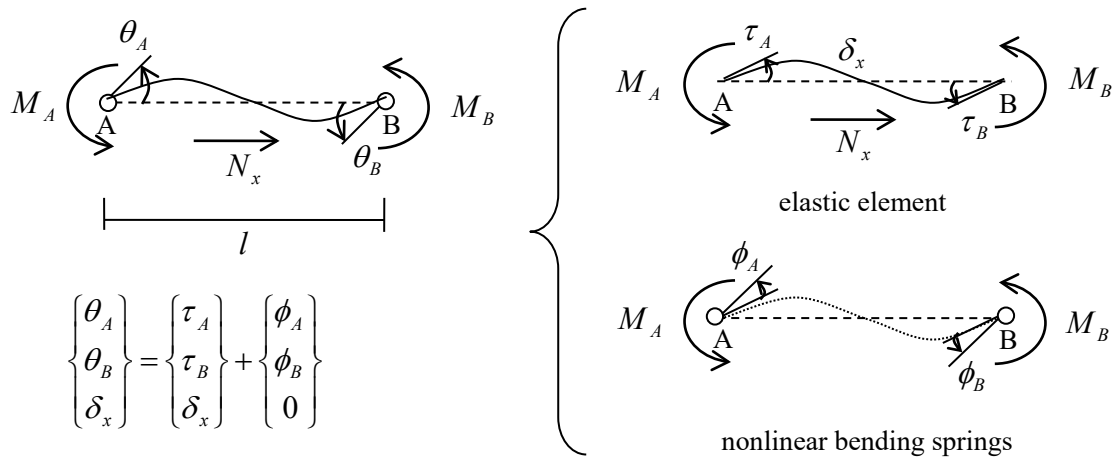


Figure 3-1-13 Element model for beam

Hysteresis model of a nonlinear bending spring is defined as the moment-rotation relationship under the anti-symmetry loading as shown in Figure 3-1-14. The initial stiffness of the nonlinear spring is supposed to be infinite, however, in numerical calculation, a large enough value is used for the stiffness.

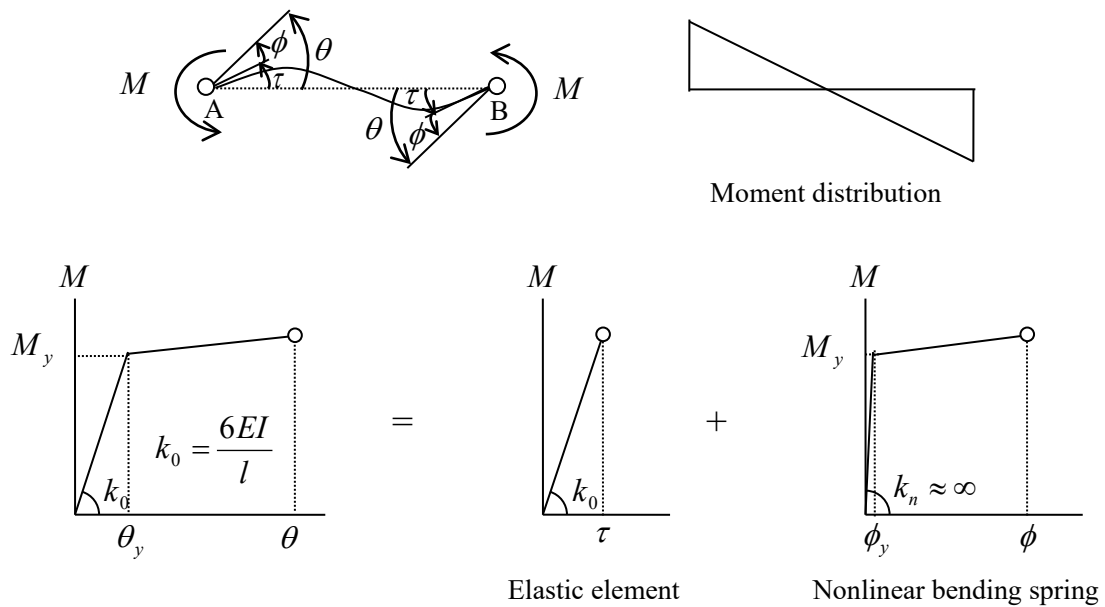


Figure 3-1-14 Moment – rotation relationship at bending spring

### Yield moment force

The yield moment,  $M_y$  is calculated as,

$$M_y = \left[ Bt_f(H - t_f) + \frac{1}{4}t_w(H - 2t_f)^2 \right] \sigma_y \quad (3-1-49)$$

where,

$\sigma_y$  : Strength of steel (N/mm<sup>2</sup>)

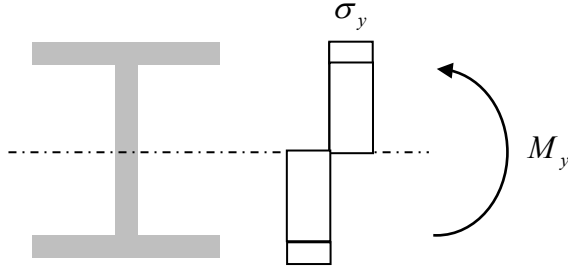


Figure 3-1-15

### Yield rotation

From Figure 3-1-14, the yield rotation of the nonlinear bending beam,  $\phi_y$ , is supposed to be zero value,

however, in STERA\_3D program, it is assumed as,

$$\phi_y = 0.001 \theta_y \quad (3-1-50)$$

where

$$\theta_y = M_y / k_0, \quad k_0 = \frac{6EI}{l}$$

### Hysteresis model

A bi-linear model is assumed for the hysteresis model.

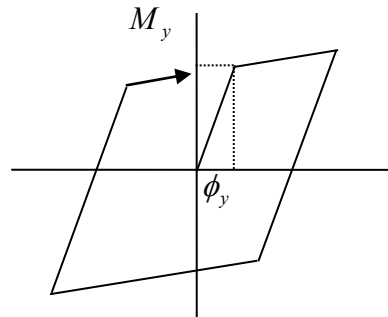


Figure 3-1-16 Hysteresis of steel

### c) Modification of initial stiffness of nonlinear springs

In numerical calculation, a large dummy value is used for the initial stiffness of the nonlinear spring to represent rigid condition. This large stiffness may cause an error for estimating the force from the displacement. One possible way to solve the problem is to reduce the initial stiffness of the nonlinear spring to be a certain value reasonable for calculation, and on the other hand, increase the stiffness of the elastic element so that the total initial stiffness of the beam element does not change from the original one. This idea is proposed by K-N Li (2004) for MS model, and can be used for nonlinear spring model also.

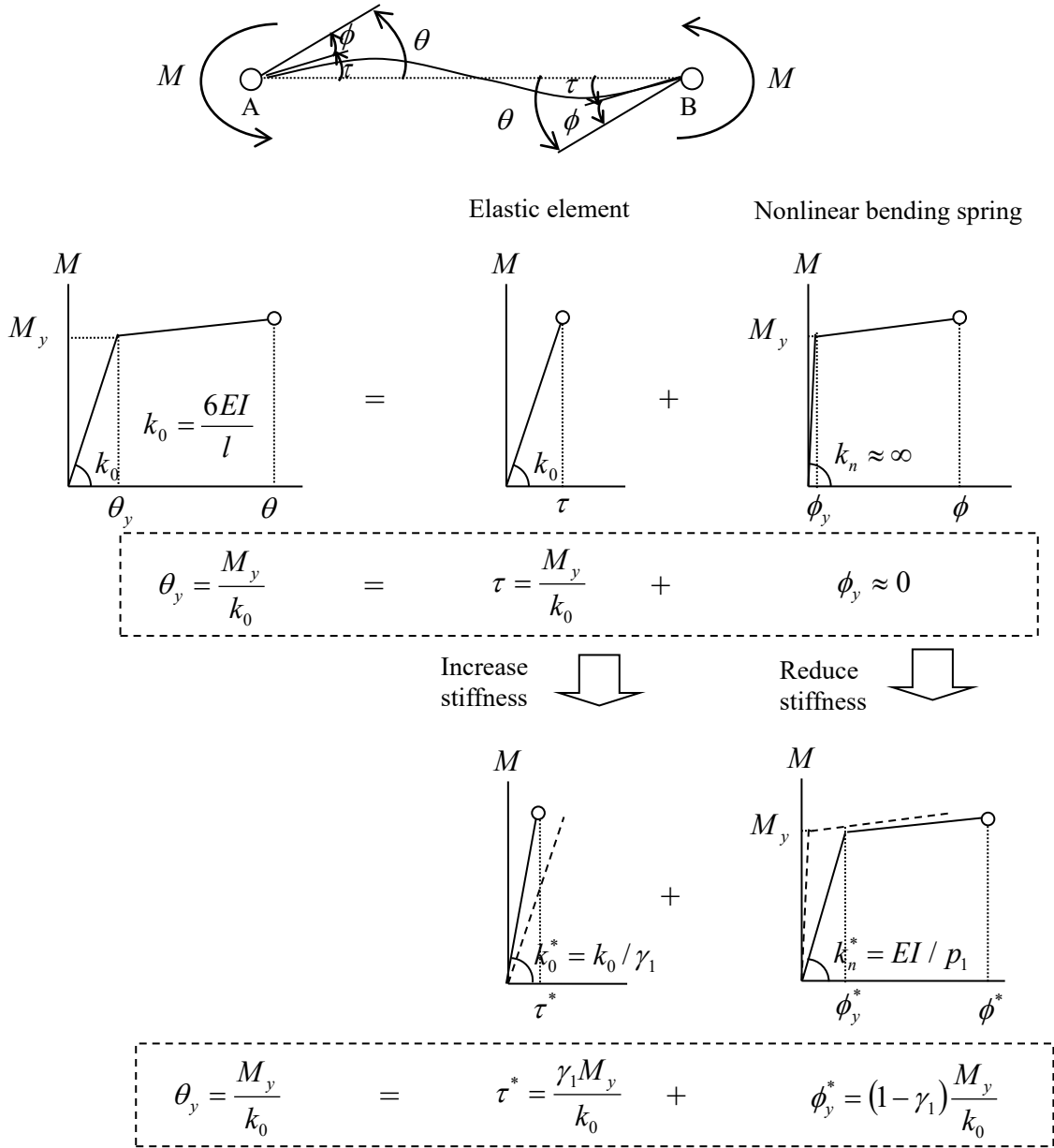


Figure 3-1-17 Modification of moment – rotation relationship

The idea is realized using flexibility reduction factors,  $\gamma_1 (< 1)$ ,  $\gamma_2 (< 1)$ , in the relationship between the displacement vector and force vector of the elastic element in Equation (2-1-1) as,

$$\begin{Bmatrix} \tau'_{yA} \\ \tau'_{yB} \\ \delta'_x \end{Bmatrix} = \begin{bmatrix} \gamma_1 \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ -\frac{l'}{6EI_y} & \gamma_2 \frac{l'}{3EI_y} & 0 \\ 0 & 0 & \frac{l'}{EA} \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ N'_x \end{Bmatrix} \quad (3-1-51)$$

It must be  $\gamma_1 \frac{l'}{3EI_y} > \frac{l'}{6EI_y}$  or  $\gamma_1 > 0.5$  and  $\gamma_2 \frac{l'}{3EI_y} > \frac{l'}{6EI_y}$  or  $\gamma_2 > 0.5$ .

Also the initial flexibility matrix of the nonlinear spring can be expressed as follows, introducing the parameters,  $p_1, p_2$  to increase the initial flexibility.

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{yB} \end{Bmatrix} = \begin{bmatrix} p_1/EI & 0 \\ 0 & p_2/EI \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{yB} \end{Bmatrix} \quad (3-1-52)$$

When  $p_1 \rightarrow 0, p_2 \rightarrow 0$ , it represents the infinite stiffness for rigid condition. Accordingly, the yield rotation will be modified as,

$$\phi_y^* = p_1 \frac{M_y}{EI} \quad (3-1-53)$$

In general, the relation between the rotation of bending spring and that of nonlinear bending spring is

$$\phi = \theta - \gamma_1 \frac{M_y}{k_0} \quad (3-1-54)$$

Making the modified flexibility matrix to be identical to the original one,

$$\begin{bmatrix} \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ & \frac{l'}{3EI_y} & 0 \\ sym. & & \frac{l'}{EA} \end{bmatrix}_{original} = \begin{bmatrix} \frac{p_1}{EI} + \gamma_1 \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ & \frac{p_2}{EI} + \gamma_2 \frac{l'}{3EI_y} & 0 \\ sym. & & \frac{l'}{EA} \end{bmatrix}_{modified} \quad (3-1-55)$$

This gives the flexivity reduction factors as:

$$\gamma_1 = 1 - \frac{3}{l'} p_1, \quad \gamma_2 = 1 - \frac{3}{l'} p_2 \quad (3-1-56)$$

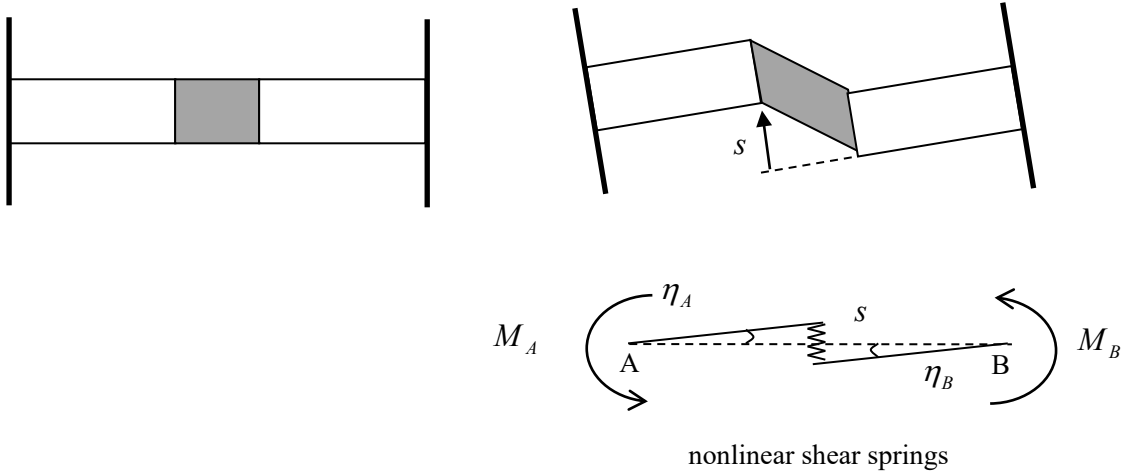
From the conditions  $\gamma_1 > 0.5$  and  $\gamma_2 > 0.5$ ,

$$p_1 < \frac{l'}{6}, \quad p_2 < \frac{l'}{6} \quad (3-1-57)$$

K-N Li (2004) calls these parameters,  $p_1, p_2$ , as “plastic zones” and recommends to be  $p_1 = p_2 = \frac{l'}{10}$ .

Then, the reduction factors will be  $\gamma_1 = \gamma_2 = 0.7$ .

**d) Shear spring for damper using low yield strength steel**



$$Q = \frac{M_A + M_B}{l}, \quad \eta_A = \eta_B = \frac{s}{l}$$

Nonlinear relationship between shear force  $Q$  and deformation  $s$  is defined.

**e) Shear spring for damper using viscoelastic material**

When velocity-dependent restoring forces are included, the constitutive equation of the member in incremental form based on the balance of forces is obtained as follows.

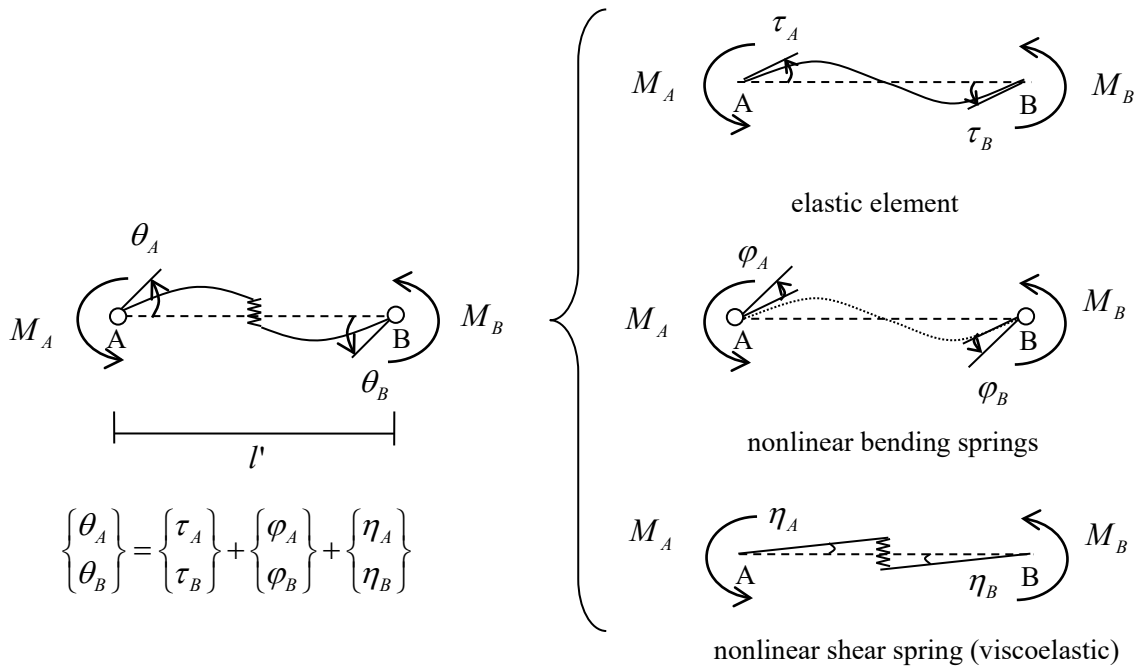


Figure 3-1-18 Element model for beam

The relationship between the displacement vector and force vector of the elastic element is expressed as follows:

$$\begin{Bmatrix} \tau_A \\ \tau_B \end{Bmatrix} = \begin{bmatrix} \frac{l}{3EI} & -\frac{l}{6EI} \\ -\frac{l}{6EI} & \frac{l}{3EI} \end{bmatrix} \begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = [F_e] \begin{Bmatrix} M_A \\ M_B \end{Bmatrix} \quad (3-1-58)$$

or

$$\begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = [K_e] \begin{Bmatrix} \tau_A \\ \tau_B \end{Bmatrix}, \quad [K_e] = [F_e]^{-1} = \begin{bmatrix} \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix} \quad (3-1-59)$$

where,  $E$ ,  $I$ ,  $A$  and  $l$  are the modulus of elasticity, the moment of inertia of the cross-sectional area, the cross-sectional area and the length of the element.

The relationship between the displacement vector and force vector of the nonlinear bending springs is,

$$\begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = \begin{Bmatrix} m(\varphi_A) \\ m(\varphi_B) \end{Bmatrix} \quad (3-1-60)$$

where,  $m(\varphi)$  is the nonlinear hysteresis model of the bending spring.

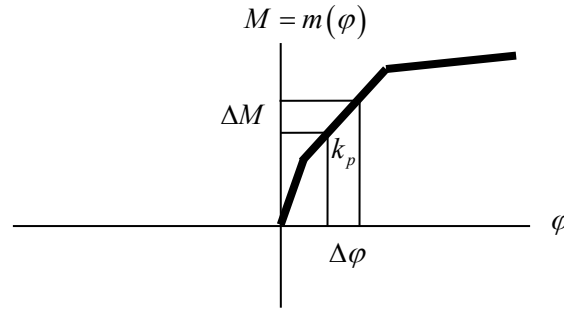


Figure 2 Nonlinear hysteresis for bending springs

In incremental form, the relationship can be linearized as

$$\begin{Bmatrix} \Delta M_A \\ \Delta M_B \end{Bmatrix} = \begin{bmatrix} k_{pA} & 0 \\ 0 & k_{pB} \end{bmatrix} \begin{Bmatrix} \Delta \varphi_A \\ \Delta \varphi_B \end{Bmatrix} = [K_p] \begin{Bmatrix} \Delta \varphi_A \\ \Delta \varphi_B \end{Bmatrix} \quad (3-1-61)$$

or

$$\begin{Bmatrix} \Delta \varphi_A \\ \Delta \varphi_B \end{Bmatrix} = \begin{bmatrix} f_{pA} & 0 \\ 0 & f_{pB} \end{bmatrix} \begin{Bmatrix} \Delta M_A \\ \Delta M_B \end{Bmatrix} = [F_p] \begin{Bmatrix} \Delta M_A \\ \Delta M_B \end{Bmatrix} \quad (3-1-62)$$

where,  $k_{pA} = 1/f_{pA}$  and  $k_{pB} = 1/f_{pB}$  are the stiffness of nonlinear bending springs in incremental range at both ends of the element.

From the relationship between shear force and moment,

$$Q = \begin{bmatrix} 1/l & 1/l \end{bmatrix} \begin{Bmatrix} M_A \\ M_B \end{Bmatrix} \quad (3-1-63)$$

The end rotational displacement due to shear deformation is obtained as,

$$\begin{Bmatrix} \eta_A \\ \eta_B \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \eta = \begin{bmatrix} 1/l \\ 1/l \end{bmatrix} \delta, \quad \eta = \frac{\delta}{l} \quad (3-1-64)$$

where,  $\delta$  is the shear deformation of the damper.

For the viscoelastic damper, the shear force is defined as

$$Q = c_s \dot{\delta} + k_s \delta \quad (3-1-65)$$

where,  $c_s$  is the damping coefficient and  $k_s$  is the stiffness of the damper.

The displacement vector of the beam element is obtained as the sum of the above three displacement vectors.

$$\begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix} = \begin{Bmatrix} \tau_A \\ \tau_B \end{Bmatrix} + \begin{Bmatrix} \varphi_A \\ \varphi_B \end{Bmatrix} + \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \delta \quad (3-1-66)$$

In incremental form

$$\begin{Bmatrix} \Delta \theta_A \\ \Delta \theta_B \end{Bmatrix} = \begin{Bmatrix} \Delta \tau_A \\ \Delta \tau_B \end{Bmatrix} + \begin{Bmatrix} \Delta \varphi_A \\ \Delta \varphi_B \end{Bmatrix} + \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \Delta \delta \quad (3-1-67)$$

From the above equations,

$$\begin{Bmatrix} \Delta M_A \\ \Delta M_B \end{Bmatrix} = [K_e] \begin{Bmatrix} \Delta \tau_A \\ \Delta \tau_B \end{Bmatrix} = [K_p] \begin{Bmatrix} \Delta \varphi_A \\ \Delta \varphi_B \end{Bmatrix} \quad (3-1-68)$$

Therefore,

$$\begin{Bmatrix} \Delta \varphi_A \\ \Delta \varphi_B \end{Bmatrix} = [K_p]^{-1} [K_e] \begin{Bmatrix} \Delta \tau_A \\ \Delta \tau_B \end{Bmatrix} = [F_p] [K_e] \begin{Bmatrix} \Delta \tau_A \\ \Delta \tau_B \end{Bmatrix} \quad (3-1-69)$$

Also,

$$\begin{Bmatrix} \Delta \theta_A \\ \Delta \theta_B \end{Bmatrix} = \begin{Bmatrix} \Delta \tau_A \\ \Delta \tau_B \end{Bmatrix} + \begin{Bmatrix} \Delta \varphi_A \\ \Delta \varphi_B \end{Bmatrix} + \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \Delta \delta = ([I] + [F_p] [K_e]) \begin{Bmatrix} \Delta \tau_A \\ \Delta \tau_B \end{Bmatrix} + \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \Delta \delta \quad (3-1-70)$$

Therefore

$$\begin{Bmatrix} \Delta \tau_A \\ \Delta \tau_B \end{Bmatrix} = ([I] + [F_p] [K_e])^{-1} \left( \begin{Bmatrix} \Delta \theta_A \\ \Delta \theta_B \end{Bmatrix} - \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \Delta \delta \right) \quad (3-1-71)$$

The equation of viscoelastic damper at time  $t$  is

$$Q_t = c_s \dot{\delta}_t + k_s \delta_t \quad (3-1-72)$$

Substitute the following equations,

$$\begin{aligned} Q_t &= Q_{t-1} + \Delta Q \\ \dot{\delta}_t &= \frac{\Delta \delta}{\Delta t} \\ \delta_t &= \delta_{t-1} + \Delta \delta \end{aligned} \quad (3-1-73)$$

Then

$$\Delta Q = c_s \frac{\Delta \delta}{\Delta t} + k_s (\delta_{t-1} + \Delta \delta) - Q_{t-1} \quad (3-1-74)$$

On the other hand,

$$\begin{aligned} \Delta Q &= \begin{bmatrix} 1/l & 1/l \end{bmatrix} \begin{Bmatrix} \Delta M_A \\ \Delta M_B \end{Bmatrix} = \begin{bmatrix} 1/l & 1/l \end{bmatrix} [K_e] \begin{Bmatrix} \Delta \tau_A \\ \Delta \tau_B \end{Bmatrix} \\ &= \begin{bmatrix} 1/l & 1/l \end{bmatrix} [K_e] \left( [I] + [F_p] [K_e] \right)^{-1} \left( \begin{Bmatrix} \Delta \theta_A \\ \Delta \theta_B \end{Bmatrix} - \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \Delta \delta \right) \\ &= [R] \left( \begin{Bmatrix} \Delta \theta_A \\ \Delta \theta_B \end{Bmatrix} - \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \Delta \delta \right), \quad [R] = \begin{bmatrix} 1/l & 1/l \end{bmatrix} [K_e] \left( [I] + [F_p] [K_e] \right)^{-1} \end{aligned} \quad (3-1-75)$$

Since the two equations are equal,

$$\begin{aligned} \Delta Q &= c_s \frac{\Delta \delta}{\Delta t} + k_s (\delta_{t-1} + \Delta \delta) - Q_{t-1} = [R] \begin{Bmatrix} \Delta \theta_A \\ \Delta \theta_B \end{Bmatrix} - [R] \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \Delta \delta \\ \left( \frac{c_s}{\Delta t} + k_s + [R] \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix} \right) \Delta \delta &= Q_{t-1} - k_s \delta_{t-1} + [R] \begin{Bmatrix} \Delta \theta_A \\ \Delta \theta_B \end{Bmatrix} \\ \Delta \delta &= \frac{Q_{t-1} - k_s \delta_{t-1} + [R] \begin{Bmatrix} \Delta \theta_A \\ \Delta \theta_B \end{Bmatrix}}{\frac{c_s}{\Delta t} + k_s + [R] \begin{Bmatrix} 1/l \\ 1/l \end{Bmatrix}} \end{aligned} \quad (3-1-76)$$

Here are the steps to get the restoring force from the deformation:

Step 1. Obtain nodal deformation  $\begin{Bmatrix} \Delta\theta_A \\ \Delta\theta_B \end{Bmatrix}$

Step 2. Calculate  $\Delta\delta$ , update  $\delta_t$  and calculate the damper force  $Q_t$

Step 3. Calculate  $\begin{Bmatrix} \Delta\tau_A \\ \Delta\tau_B \end{Bmatrix}$  and  $\begin{Bmatrix} \Delta\varphi_A \\ \Delta\varphi_B \end{Bmatrix}$

Step 4. From the nonlinear hysteresis, get nodal force  $\begin{Bmatrix} M_A \\ M_B \end{Bmatrix}$  from  $\begin{Bmatrix} \Delta\varphi_A \\ \Delta\varphi_B \end{Bmatrix}$

### 3.1.3 SRC Beam

#### a) Section properties

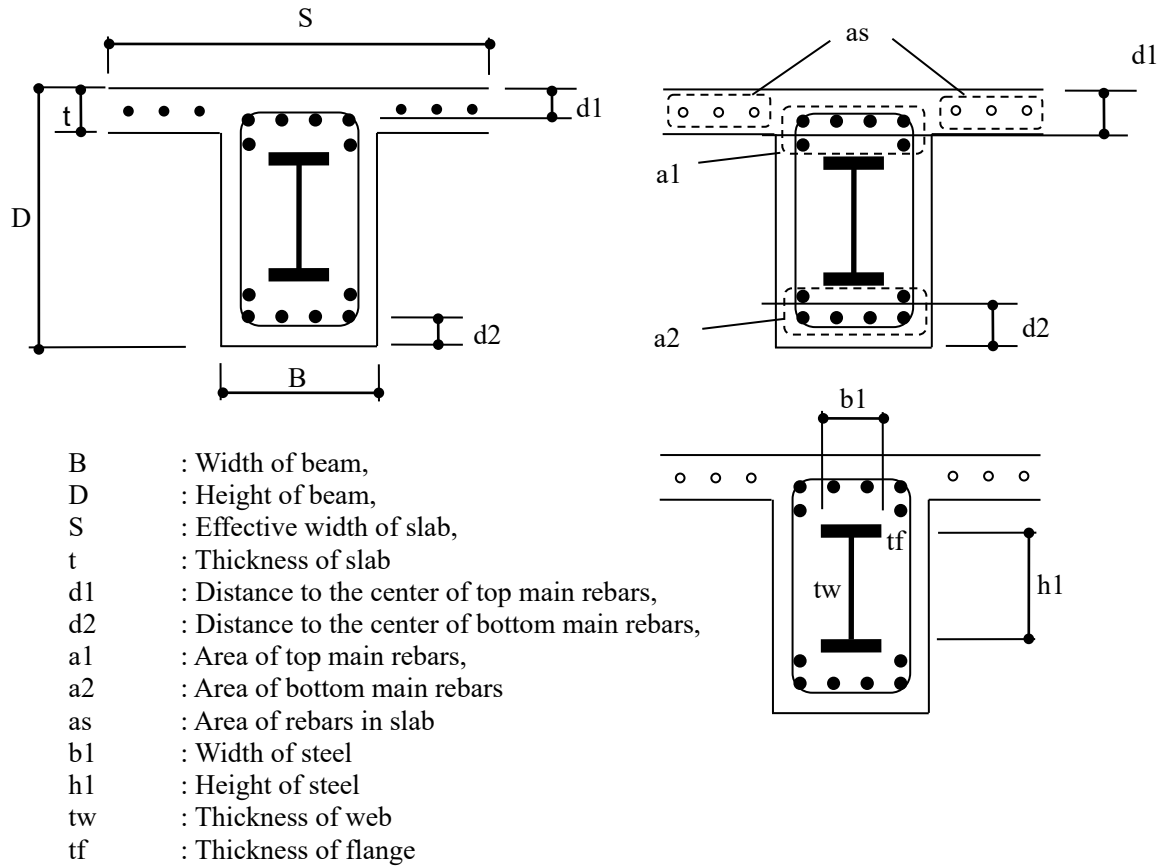


Figure 3-1-18 SRC Beam Section

Area of section to calculate axial deformation

$$A_N = BD + (S - B)t + (n_E - 1)(a_1 + a_2 + a_s + a_{ST}) \quad (3-1-58)$$

where,

$$n_E = E_s / E_c \quad : \text{Ratio of Young's modulus between steel } (E_s) \text{ and concrete } (E_c)$$

$$a_{ST} = 2(b_1 - t_w)t_f + h_1 t_w \quad : \text{Area of steel}$$

Area of section to calculate shear deformation

$$A_S = BD \quad (3-1-59)$$

Moment of inertia around the center of the section

$$\begin{aligned}
 I_e = & \frac{BD^3}{12} + \frac{(S - B)t^3}{12} + BD \left( g - \frac{D}{2} \right)^2 + (S - B)t \left( D - \frac{t}{2} - g \right)^2 + \\
 & (n_E - 1)a_1(d_1 - g)^2 + (n_E - 1)a_2(D - d_2 - g)^2 + (n_E - 1)a_s \left( D - \frac{t}{2} - g \right)^2 + \\
 & (n_E - 1) \frac{b_1 h_1^3 - (b_1 - t_w)(h_1 - 2t_f)^3}{12}
 \end{aligned} \quad (3-1-60)$$

where,  $g$  is the center of beam section calculated by

$$g = \frac{BD^2/2 + (S-B)t(D-t/2) + (n_E - 1)(a_1d_1 + a_2(D-d_2) + a_s(D-t/2) + a_{sT}D/2)}{A_N} \quad (3-1-61)$$

### b) Nonlinear bending spring

Hysteresis model of a nonlinear bending spring is the same as RC beam.

#### Crack moment force

For reinforced concrete elements, the crack moment,  $M_c$  is calculated as,

$$M_{c1} = 0.56\sqrt{\sigma_B}Z_{e1}, \quad Z_{e1} = I_e / g \quad \text{when tension in top main rebars} \quad (3-1-62)$$

$$M_{c2} = 0.56\sqrt{\sigma_B}Z_{e2}, \quad Z_{e2} = I_e / (D - g) \quad \text{when tension in bottom main rebars} \quad (3-1-63)$$

where,

$\sigma_B$  : Compression strength of concrete (N/mm<sup>2</sup>)

$Z_{e1}, Z_{e2}$  : Section modulus

#### Yield moment force

The yield moment,  $M_y$  is calculated as,

$$M_y = M_{y1,2,RC} + M_{y,S} \quad (3-1-64)$$

where

$$M_{y1,2,RC} : \text{Yield moment of reinforced concrete} \quad (3-1-65)$$

$$M_{y1,RC} = 0.9a_1\sigma_y(D-d_1) + 0.9a_s\sigma_y(D-t/2) \quad \text{when tension in top main rebars}$$

$$M_{y2,RC} = 0.9a_2\sigma_y(D-d_2) \quad \text{when tension in bottom main rebars}$$

where,

$\sigma_y$  : Strength of rebar (N/mm<sup>2</sup>)

$$M_{y,S} = \left[ b_1t_f(h_1 - t_f) + \frac{1}{4}t_w(h_1 - 2t_f)^2 \right] \sigma_{y,S} : \text{Yield moment of steel} \quad (3-1-66)$$

where,

$\sigma_{y,S}$  : Strength of steel (N/mm<sup>2</sup>)

## Appendix 3.1:

### A-1. Hysteresis of Degrading Trilinear Slip Model

In OPTION menu in Beam Editor, you can control the shape of hysteresis loop.

The image shows two software windows. The 'Beam Editor' window on the left contains sections for 'BEAM' properties, including a list of beam types (B1 to B18), size dimensions (B, D, S, d1, d2), main reinforcement bar details (TOP, BOTTOM, SD), shear reinforcement bar details, slab reinforcement details, and concrete strength (Fc). The 'Beam Option Editor' window on the right, titled 'BEAM OPTION', contains eight numbered options for controlling the hysteresis loop, each with a numerical input field and a default value.

Option	Parameter	Default Value
1.	Amplification Factor for Steel Strength [0,2]	1.1
2.	Rs : Effective Slab Ratio [0,0.5]	0.1
3.	R1 : Stiffness Degrading Ratio [0,1]	0.5
4.	R2 : Slip Stiffness Ratio [0,1]	0
5.	R3 : Strength Degrading Ratio [0,1]	0
6.	Ru : Ultimate Rotation Angle [0,1]	0.02
7.	Kp/Ky : Stiffness Ratio over Ry [0, 1]	0.001
8.	Ku/Ky : Stiffness Ratio over Ru [-1, 1/1000]	0.001

$R_s$  : Effective Slab Ratio

As described in Eq. (3-1-8), when tension in slab side, the yield moment of beam,  $M_{y1}$ , is

$$M_{y1} = 0.9a_2\sigma_y(D-d_2) + 0.9a_s\sigma_y(D-t/2)$$

where,  $a_s$  is the area of rebars in effective width of slab,  $S_b$ , which is defined as Eq.(3-1-17),

$$S_b = \eta_s L_s$$

$\eta_s$  ( $R_s$  in the menu) is the effective slab ratio, the default value is 0.1.

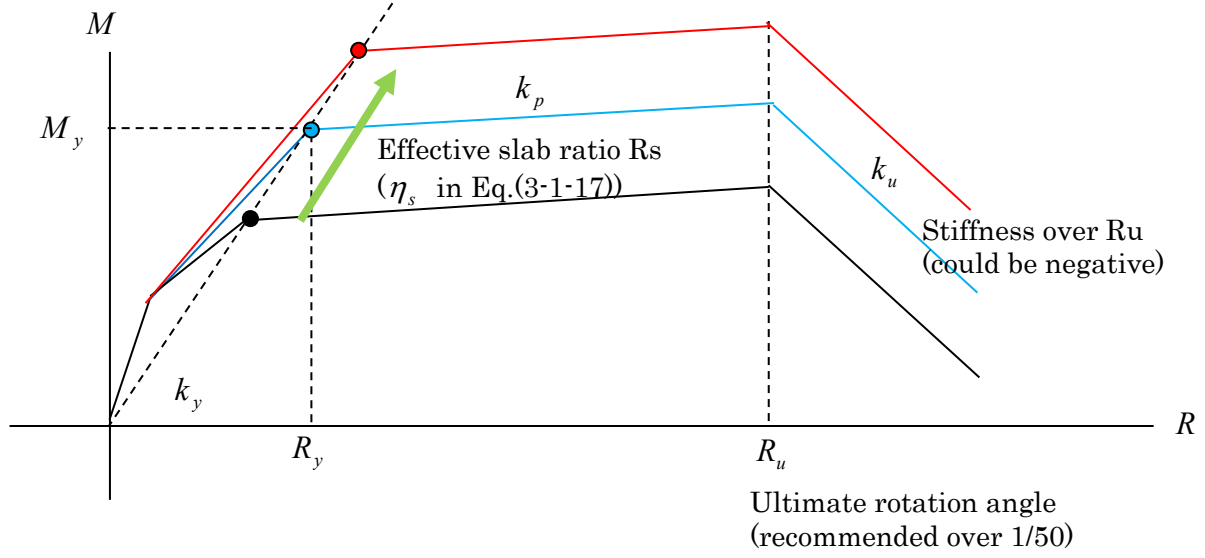
Depending on the effective slab ratio  $R_s$ , the yield moment  $M_y$  and the yield rotation  $R_y$  will change together as shown in the Figure below, since the tangential stiffness at the yield point,  $K_y$ , is assumed to be the same.

$R_u$  : Ultimate rotation angle to define the maximum moment before degradation. The default value is 1/50.

$K_p$  : The stiffness after the yield rotation angle,  $R_y$ .

$K_u$  : The stiffness after the ultimate rotation angle,  $R_u$ .

It can be the negative value to consider strength degradation, however, the default value of the ratio  $K_u / K_y$  is 1/1000 without degradation.

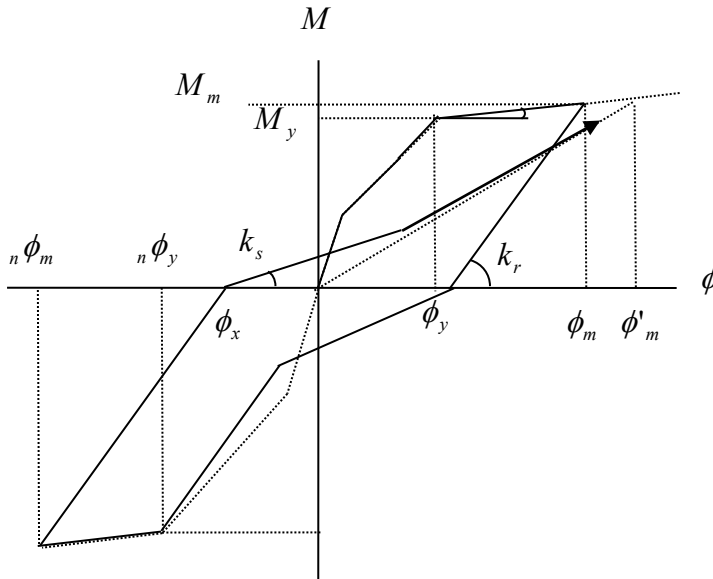


$R_1$  : stiffness degrading ratio in the trilinear hysteresis is 0.5. (0: no degradation)

$R_2$  : slip stiffness ratio in the trilinear hysteresis is 0.0 (0: no slip).

$R_3$  : strength degrading ratio in the trilinear hysteresis is 0.0.

Those parameters control the shape of hysteresis loop as described in Eqs. (3-1-18) and (3-1-19). That is,



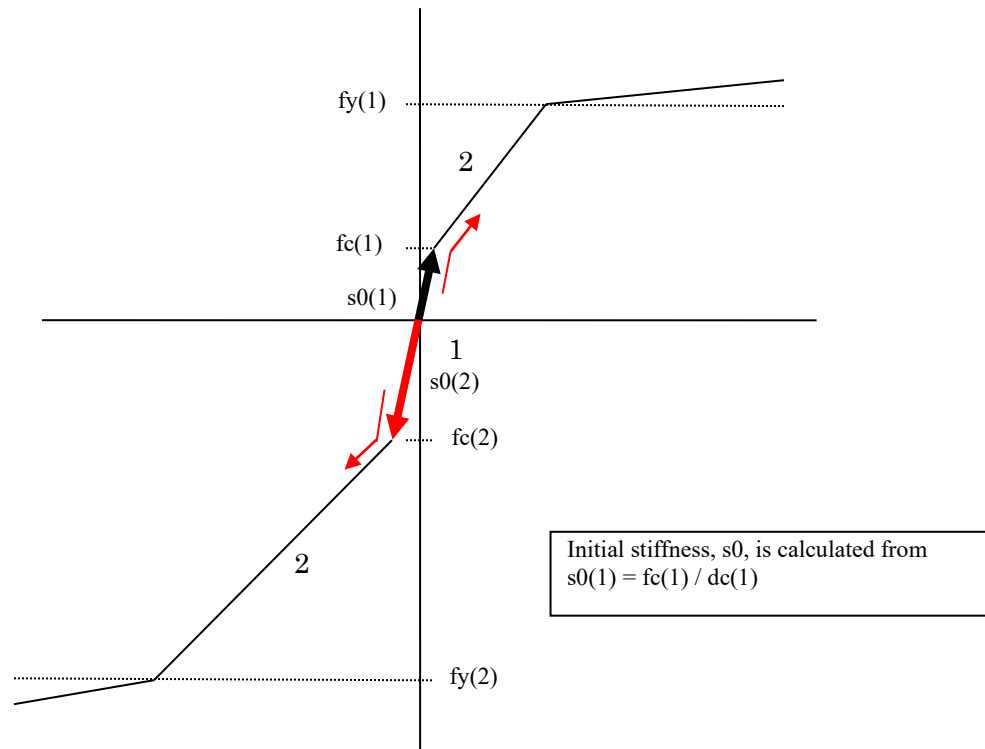
$$k_r = \left( \frac{M_y}{\phi_y} \right) \left| \frac{\phi_y}{\phi_m} \right|^\alpha \quad (\alpha = R_1)$$

$$k_s = \left( \frac{M_m}{\phi_m - \phi_x} \right) \left| \frac{\phi_y}{\phi_m} \right|^\beta \quad (\beta = R_2)$$

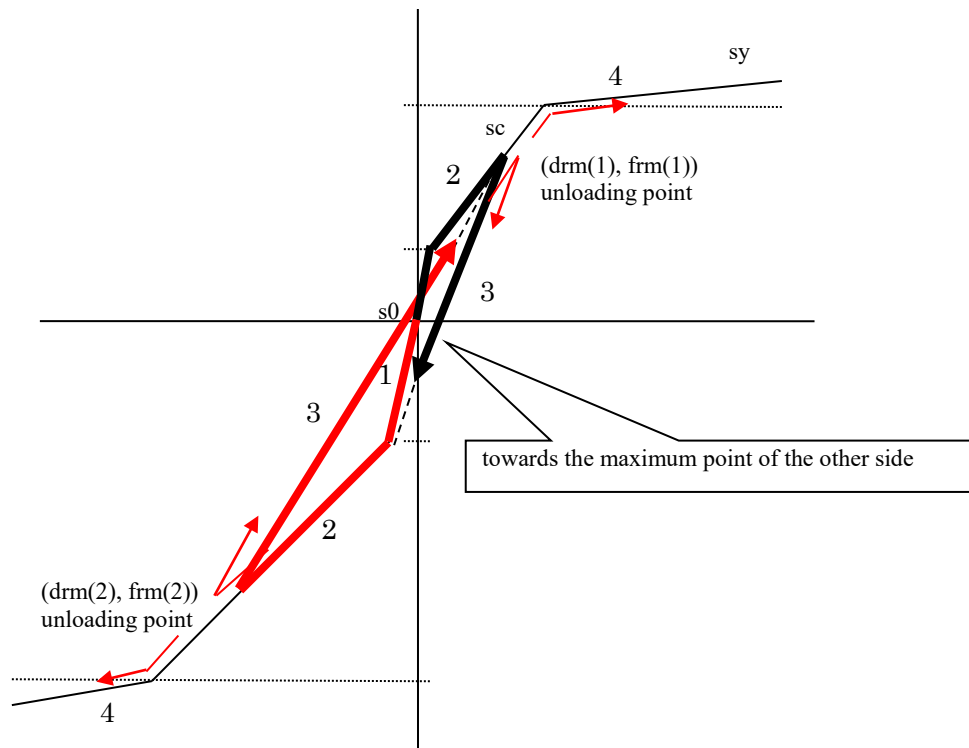
$$\phi'_m = \left( 1 + \gamma \left| \frac{n \phi_y}{n \phi_m} \right| \right) \phi_m \quad (\gamma = R_3)$$

More detail rule in the hysteresis loop is described in the following sections:

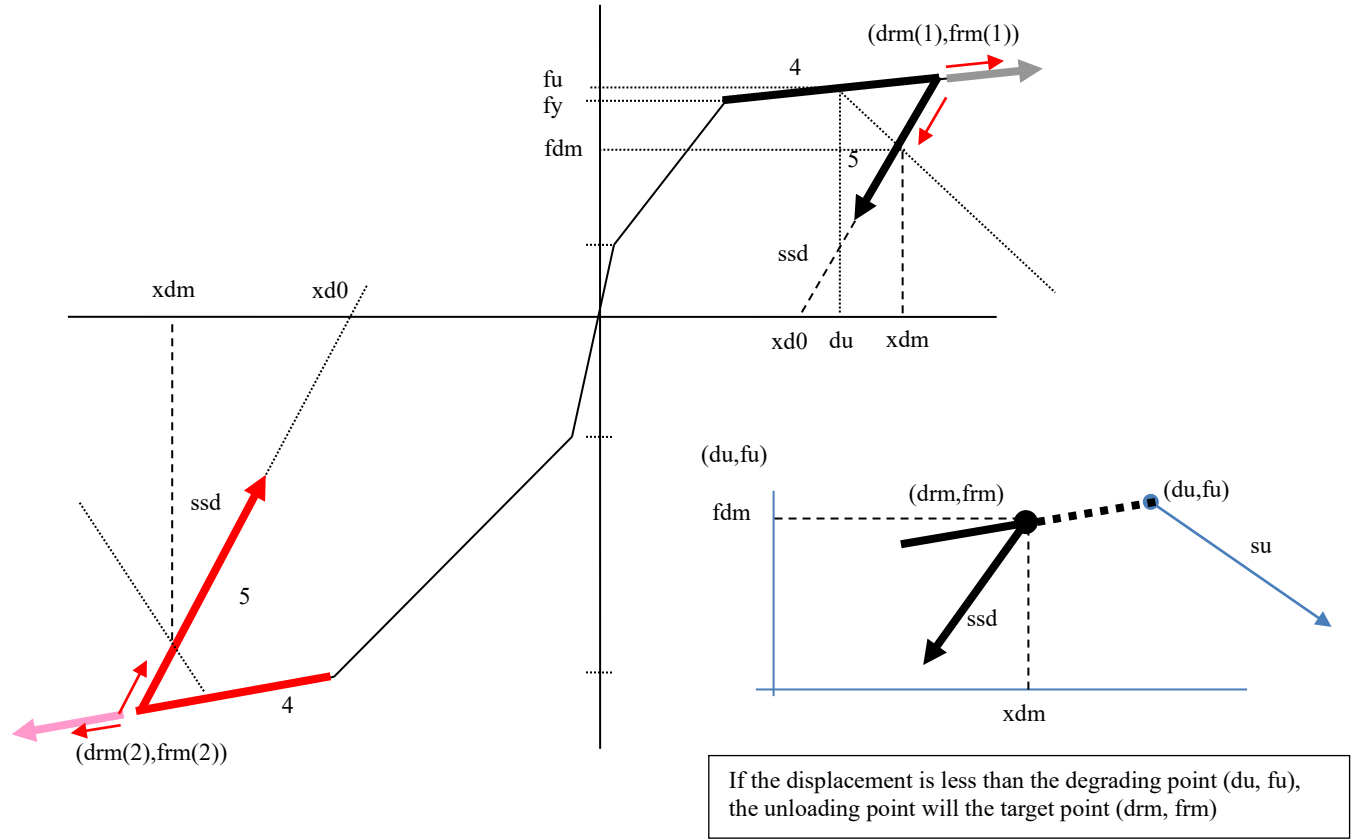
## 1. Elastic range



## 2. From crack point to yield point

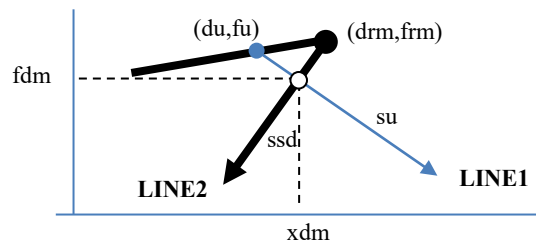


### 3. Loading on the primary curve after yielding

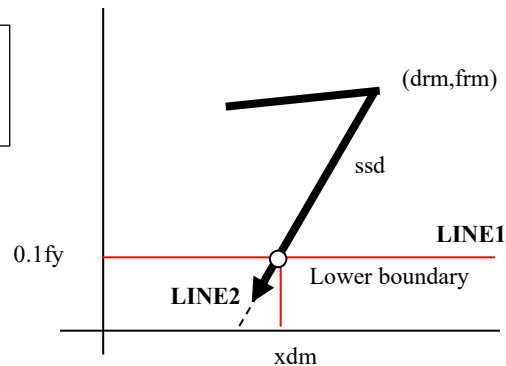


The stiffness of unloading,  $ssd$ , will be calculated from  $ssd = \left( \frac{f_y}{d_y} \right) \left| \frac{dy}{d_{rm}} \right|^\alpha$ , where  $\alpha$  is the parameter to control the stiffness degradation depending on the ductility factor,  $(d_{rm}/d_y)$ . The default value of is  $\alpha = 0.5$

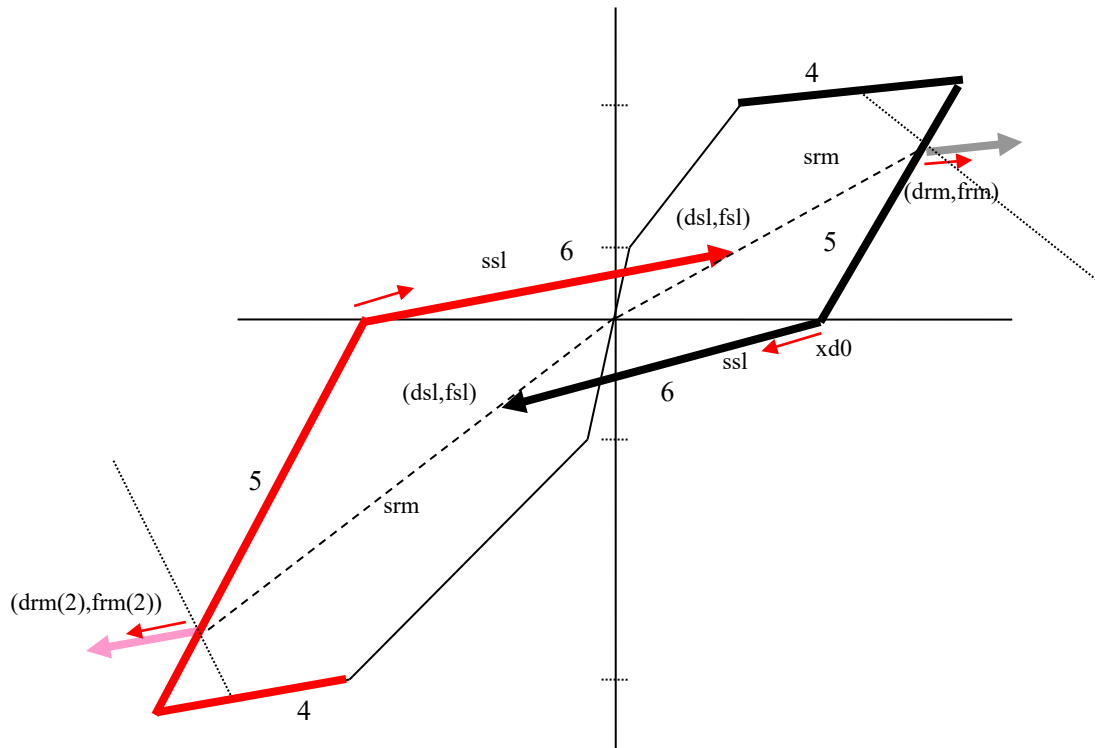
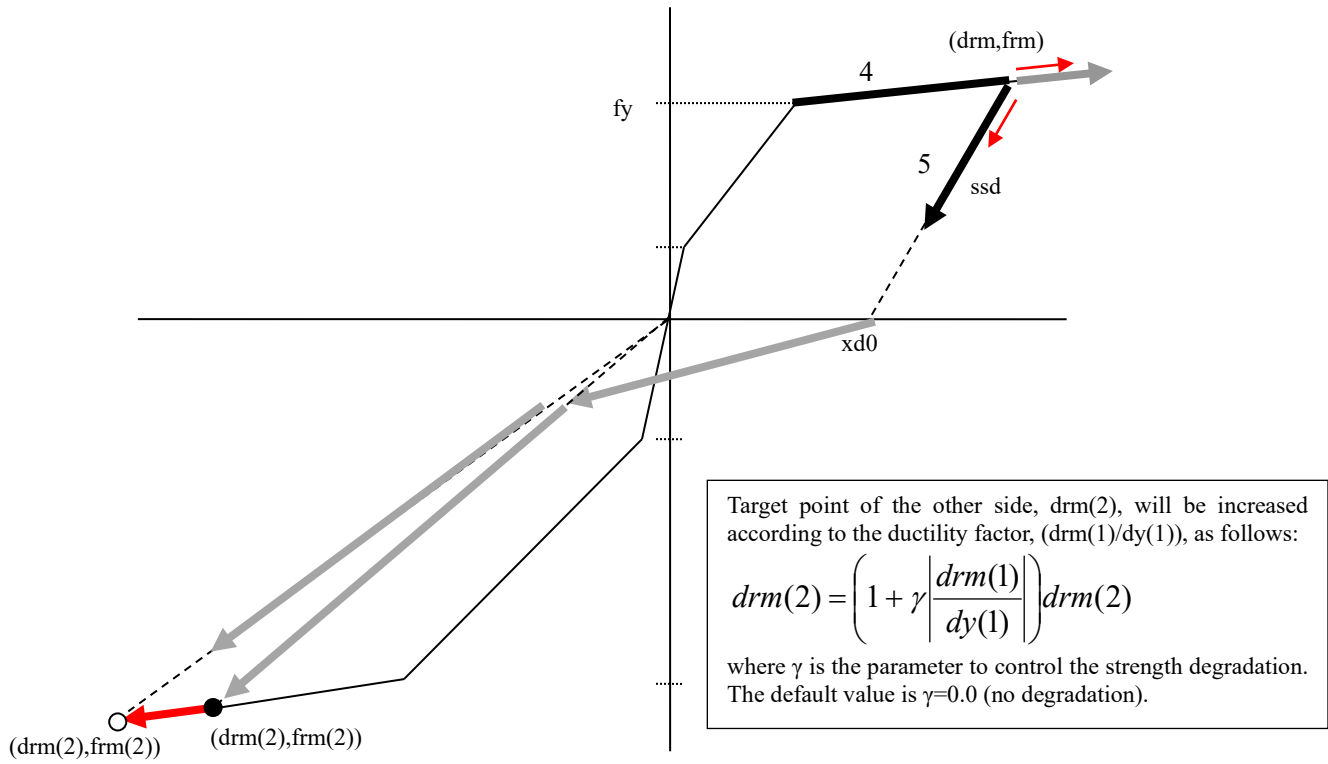
If the displacement is over the degrading point, intersection of the LINE1 (degrading line) and LINE2 (unloading line) will be the target point.



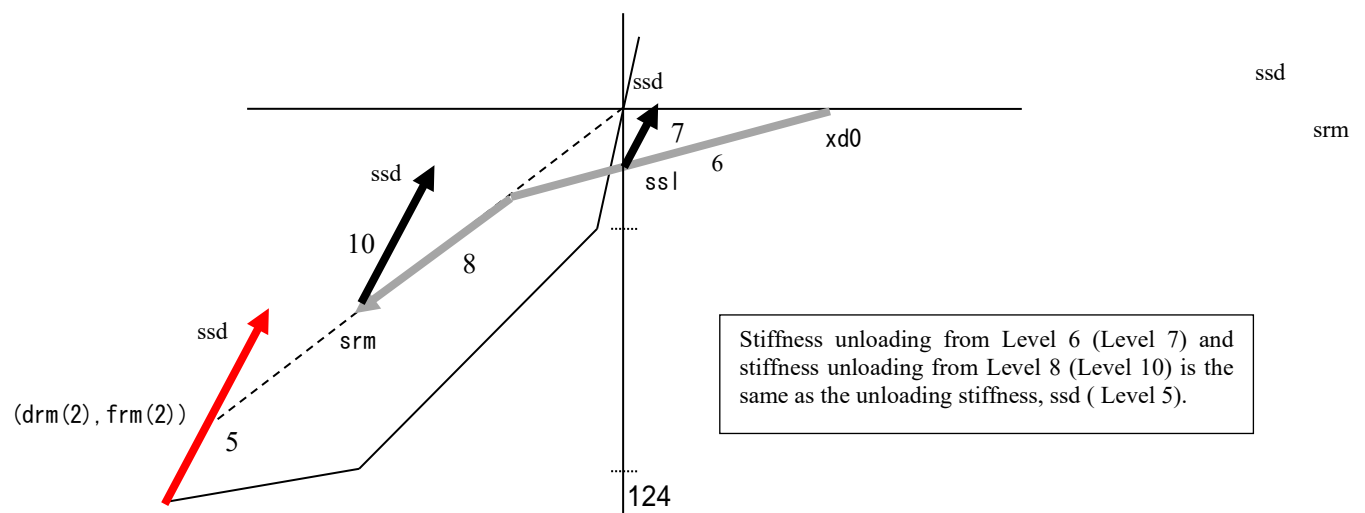
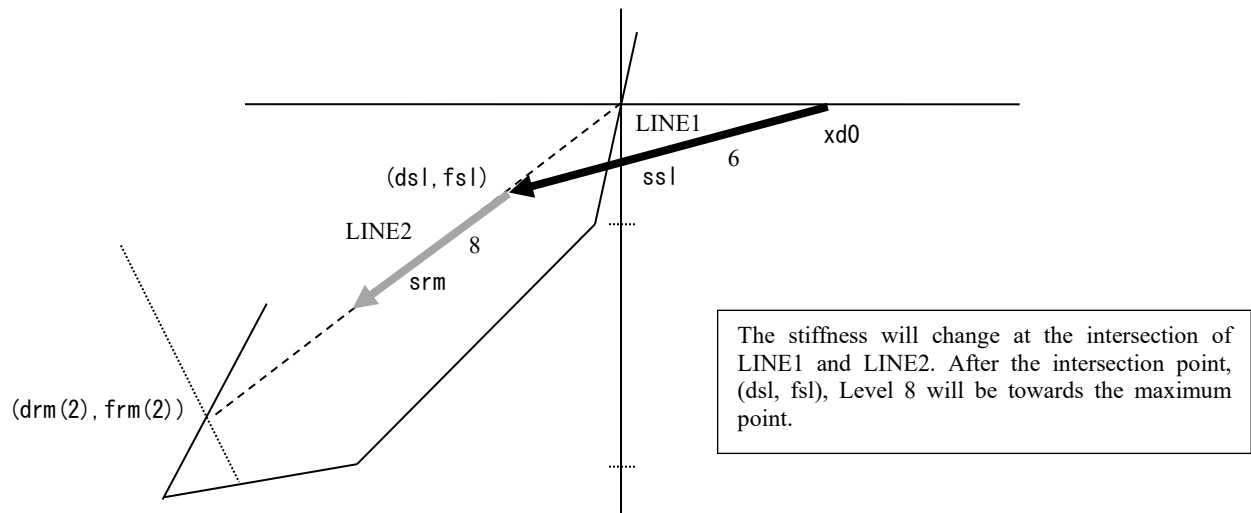
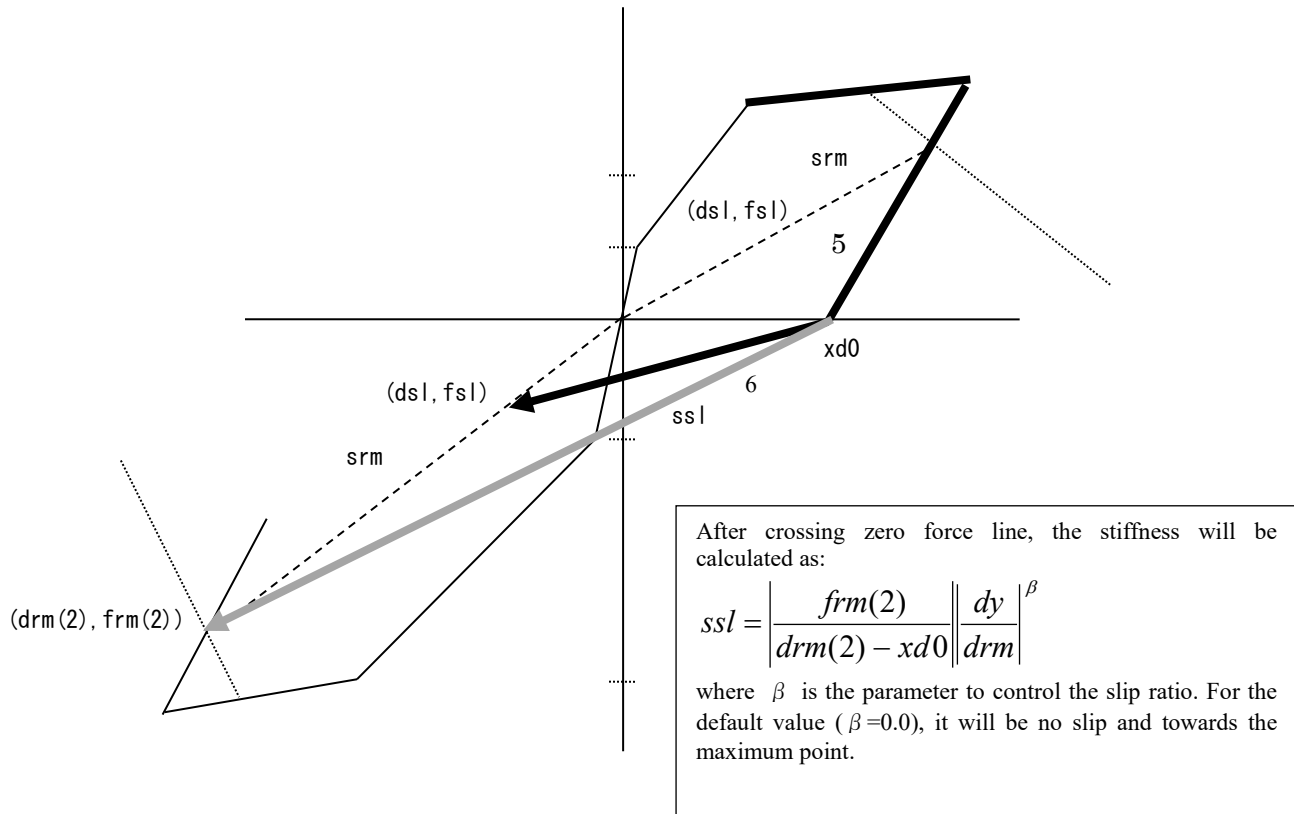
If the force is lower than  $0.1f_y$ , intersection of the LINE1 (degrading line) and LINE2 (lower boundary) will be the target point.

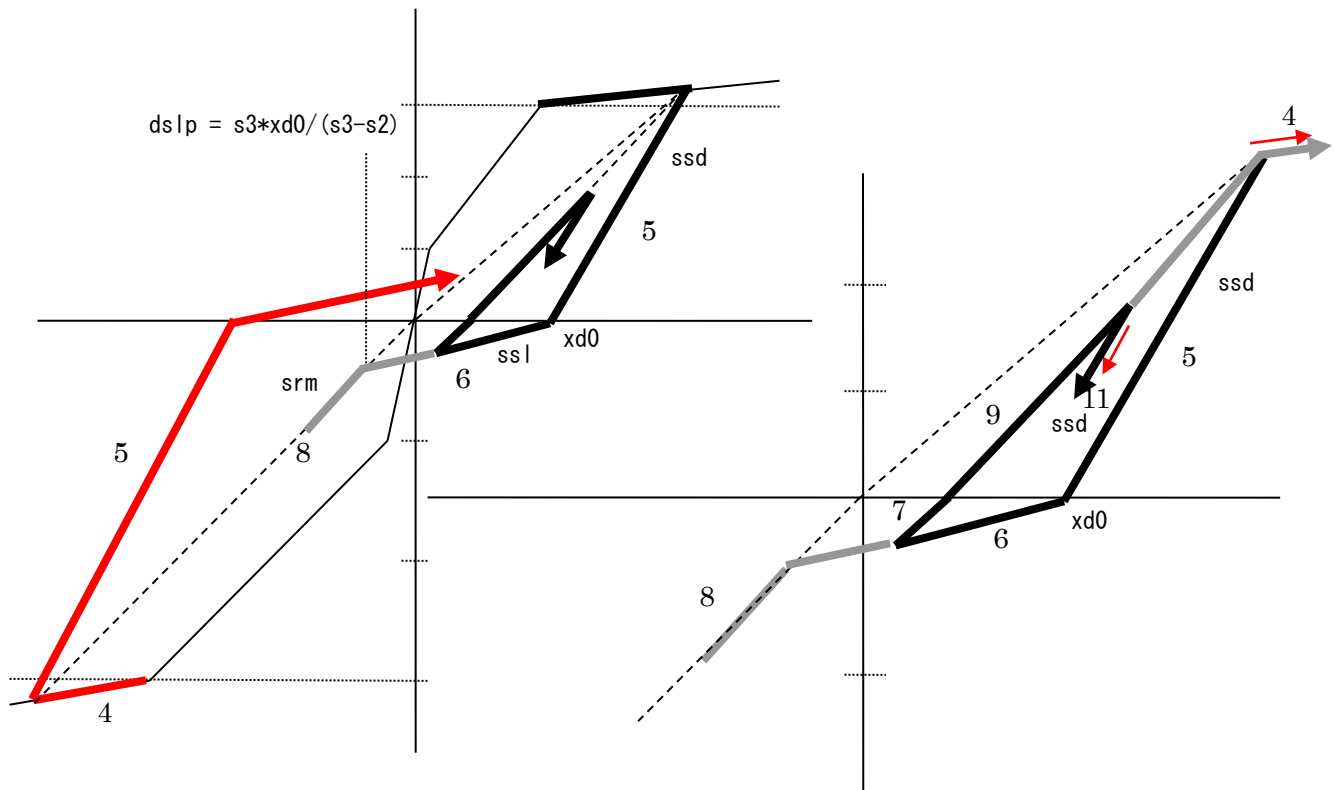


#### 4. Crossing zero force line

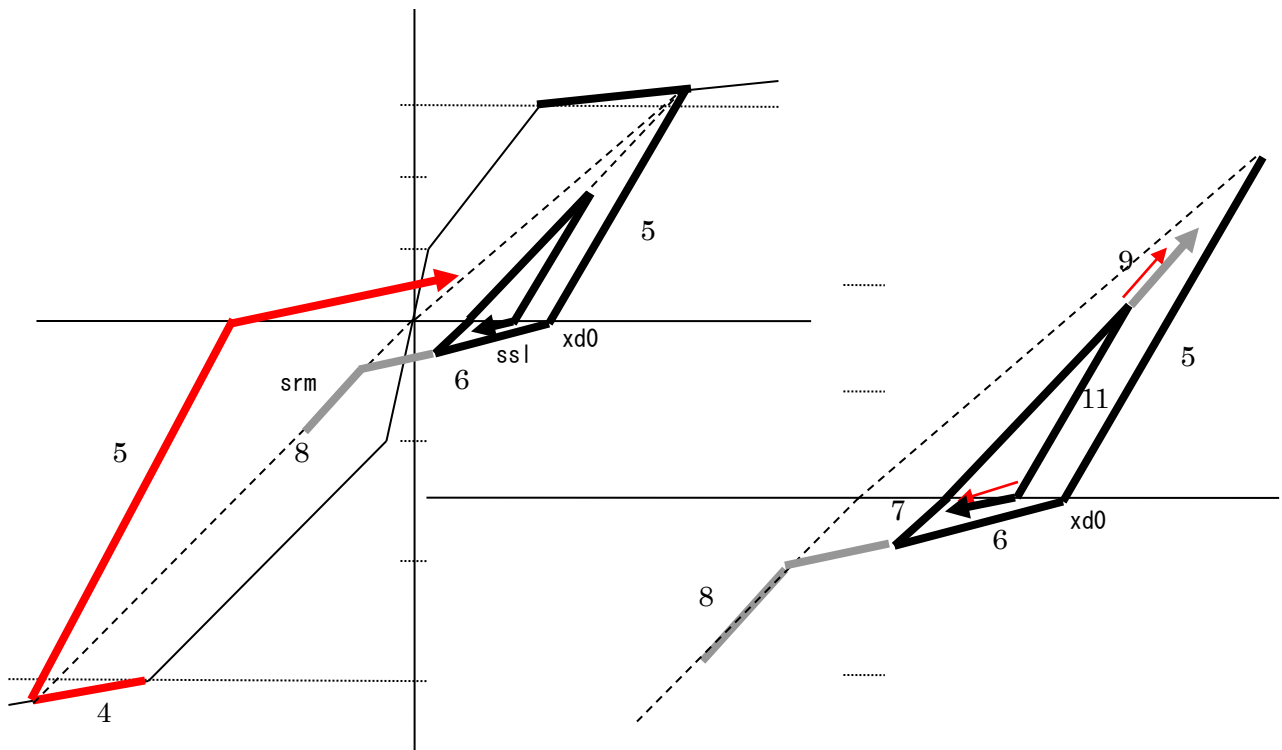


## 5. Calculation of slip point





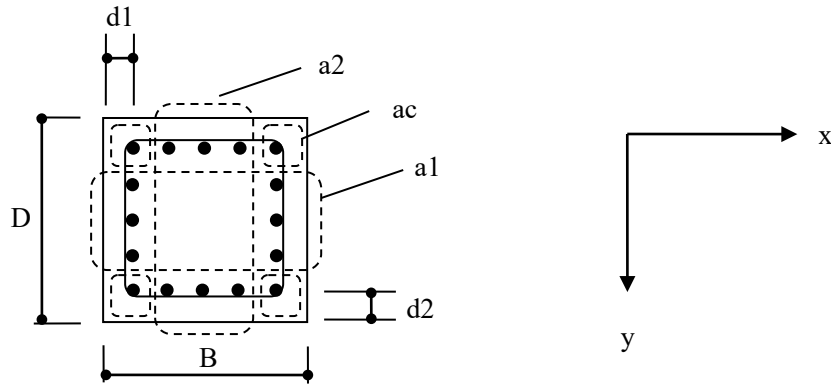
After crossing zero force line from Level 7, Level 9 will be towards the maximum point. Stiffness unloading from Level 9 (Level 11) is the same as the stiffness of ssd (Level 5).



### 3.2 Column

#### 3.2.1 RC Column

##### a) Section properties



- $B$  : Width of column,
- $D$  : Height of column,
- $d_1$  : Distance to the center of x-direction main rebars,
- $d_2$  : Distance to the center of y-direction main rebars,
- $a_1$  : Area of x-side main rebars,
- $a_2$  : Area of y-side main rebars,
- $a_c$  : Area of corner main rebars

Figure 3-2-1 RC Column Section

Area of section to calculate axial deformation

$$A_N = BD + (n_E - 1)(a_1 + a_2 + a_c) \quad (3-2-1)$$

Area of section to calculate shear deformation

$$A_S = BD / \kappa, \quad \kappa = 1.2 \quad (3-2-2)$$

Moment of inertia around the center of the section

$$I_y = \frac{DB^3}{12} + (n_E - 1)(a_c + a_1) \left( \frac{B}{2} - d_1 \right)^2 \quad (3-2-3)$$

$$I_x = \frac{BD^3}{12} + (n_E - 1)(a_c + a_2) \left( \frac{D}{2} - d_2 \right)^2 \quad (3-2-4)$$

## b) Nonlinear bending spring

Hysteresis model of nonlinear bending spring is defined as the moment-rotation relationship under the anti-symmetry loading in Figure 3-2-2. The initial stiffness of the nonlinear spring is supposed to be infinite, however, in numerical calculation, a large enough value is used for the stiffness.

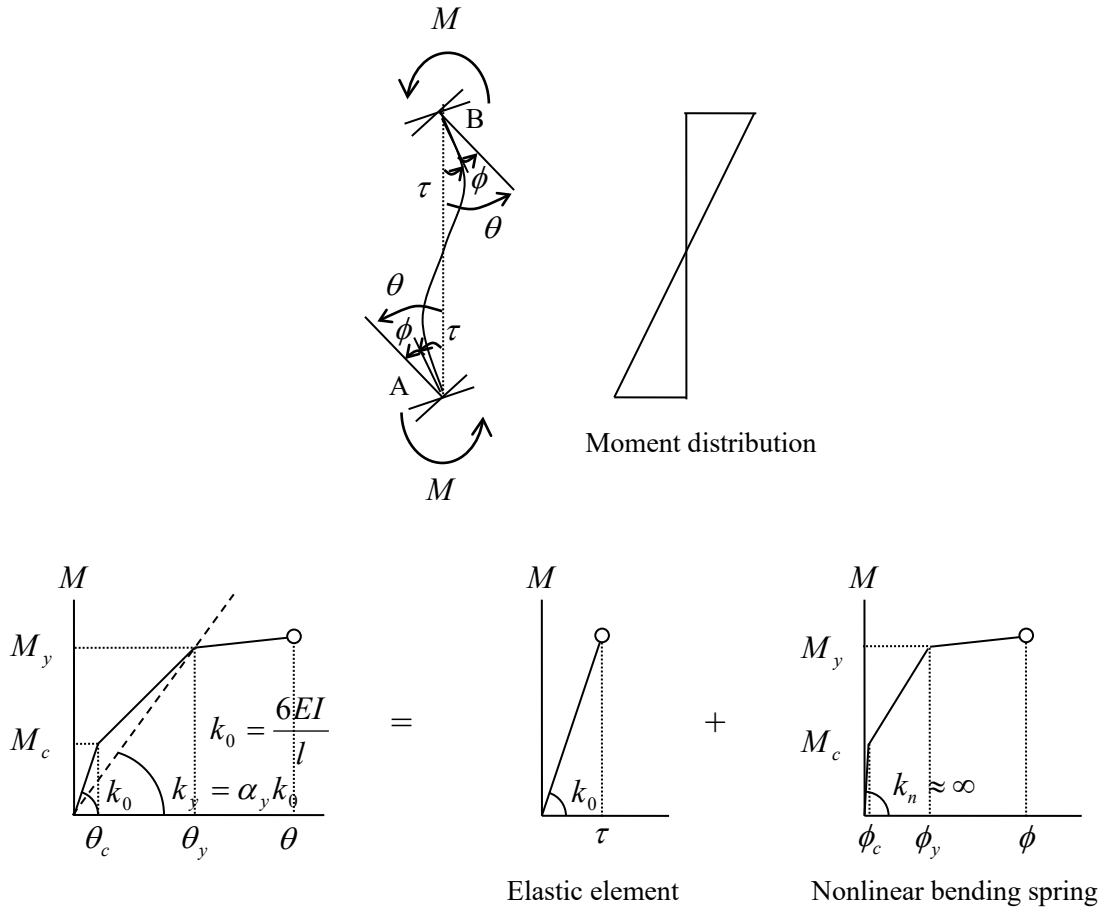


Figure 3-2-2 Moment – rotation relationship at bending spring

**The crack moment,  $M_c$**  is calculated as,

$$M_c = 0.56\sqrt{\sigma_B}Z_e + \frac{ND}{6} \quad (3-2-5)$$

where,

$\sigma_B$	:	Compression strength of concrete (N/mm <sup>2</sup> )
$Z_e$	:	Section modulus
$N$	:	Axial load

The yield moment,  $M_y$  is calculated from the following formula under the axial load,  $N$

if  $(0 < N \leq N_b)$

$$M_y = 0.8a_t\sigma_y D + 0.5ND \left(1 - \frac{N}{bD\sigma_B}\right) \quad (3-2-6)$$

if  $(N_b < N \leq N_{\max})$

$$M_y = \left(0.8a_t\sigma_y D + 0.12bD^2\sigma_B\right) \left(\frac{N_{\max} - N}{N_{\max} - N_b}\right) \quad (3-2-7)$$

where,  $N_b$  is the balance axial force,

$$N_b \approx 0.4bD\sigma_B \quad (3-2-8)$$

and  $N_{\max}$  is the maximum axial force,

$$N_{\max} \approx bD\sigma_B + A_s\sigma_y \quad (3-2-9)$$

The tangential stiffness at the yield point,  $k_y$ , is obtained from the following equation,

$$k_y = \alpha_y K_0 \quad K_0 = \frac{6EI}{l} \quad (3-2-10)$$

where,

$\alpha_y$  is the stiffness degradation factor at the yield point, which is obtained from the following empirical formulas:

$$\alpha_y = (0.043 + 1.64np_t + 0.043a/D + 0.325\eta_0)(d/D)^2, \quad (2 \leq a/D) \quad (3-2-11)$$

$$\alpha_y = (-0.0836 + 0.159a/D + 0.169\eta_0)(d/D)^2, \quad (1 \leq a/D < 2) \quad (3-2-12)$$

where,

$p_t$	:	Tensile reinforcement ratio
		$p_t = (a_c + a_1)/(2BD)$ (when tension in x-main rebars)
		$p_t = (a_c + a_2)/(2BD)$ (when tension in y-main rebars)
$a/D$	:	$\approx$ Shear span-to-depth ratio $(=l/(2D))$
$d$	:	effective depth
		$d = D-d1$ (when tension in bottom main rebars)
		$d = D-d2$ (when tension in upper main rebars)
$\eta_0 = \frac{N}{bD\sigma_B}$	:	Axial load ratio

The yield rotation of the nonlinear bending beam,  $\phi_y$ , is then obtained from,

$$\phi_y = \left( \frac{1}{\alpha_y} - 1 \right) \frac{M_y}{K_0} \quad (3-2-13)$$

**Reference:**

AIJ Standard for Structural Calculation of Reinforced Concrete Structures, Architectural Institute of Japan, 2018 (in Japanese)

**Case 1: In the case that bending springs in x and y directions are independently defined**

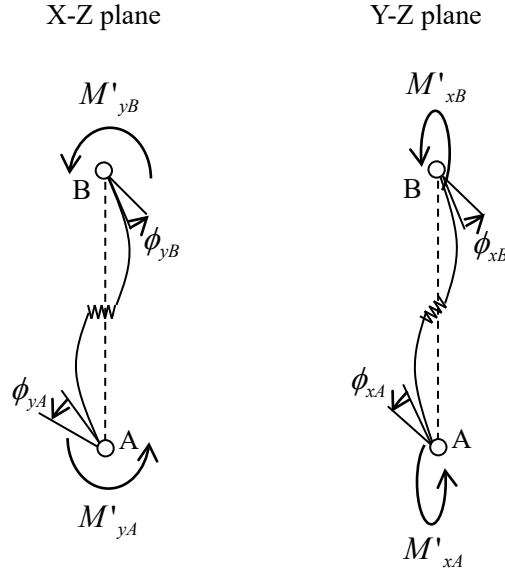


Figure 3-2-3 Element model for column

The rotational displacement vector of the nonlinear bending spring is defined independently,

$$\phi_{yA} = f_{yA} M'_{yA}, \quad \phi_{xA} = f_{xA} M'_{xA} \quad \text{at end A} \quad (3-2-14)$$

$$\phi_{yB} = f_{yB} M'_{yB}, \quad \phi_{xB} = f_{xB} M'_{xB} \quad \text{at end B} \quad (3-2-15)$$

where,  $f_{yA}$ ,  $f_{xA}$ ,  $f_{yB}$ , and  $f_{xB}$  are the flexural stiffness of nonlinear bending springs at both ends of the element, and

$$f_{yA} = 1/k_{yA}, \quad f_{xA} = 1/k_{xA}, \quad f_{yB} = 1/k_{yB}, \quad f_{xB} = 1/k_{xB} \quad (3-2-16)$$

The rotational displacement vector of the nonlinear bending springs will be

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \\ \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \\ M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} = \begin{bmatrix} f_{yA} & & & & & \\ & f_{xA} & & & & \\ & & 0 & & & \\ & & & f_{yB} & & \\ & & & & f_{xB} & \\ & & & & & 0 \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \\ M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} \quad (3-2-17)$$

The hysteresis model for  $M - \phi$  relationship is the degrading tri-linear slip model as used for the hysteresis model of the bending springs of the RC beam.

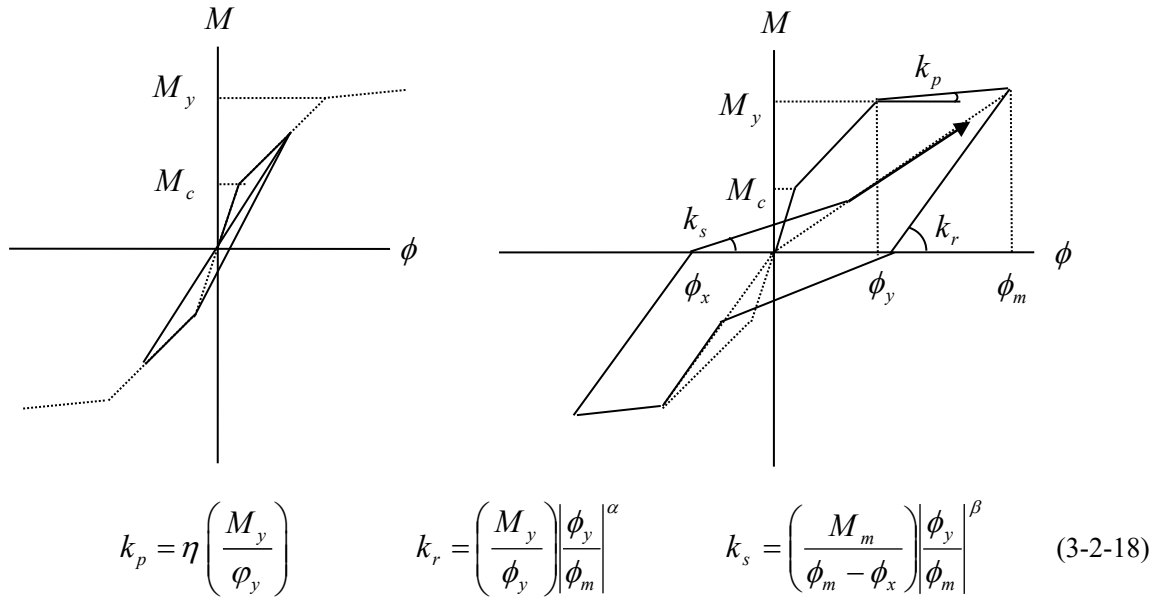


Figure 3-2-4 Degrading Tri-linear Slip Model  
( $\alpha=0.5$ ,  $\beta=0.0$  and  $\eta=0.001$  as default values)

**Case 2: In the case that nonlinear interaction between moment and axial components is considered**

To consider nonlinear interaction among  $M_x - M_y - N_z$ , the nonlinear bending spring at the member end is constructed from the nonlinear vertical springs arranged in the member section as shown in Figure 3-2-4.

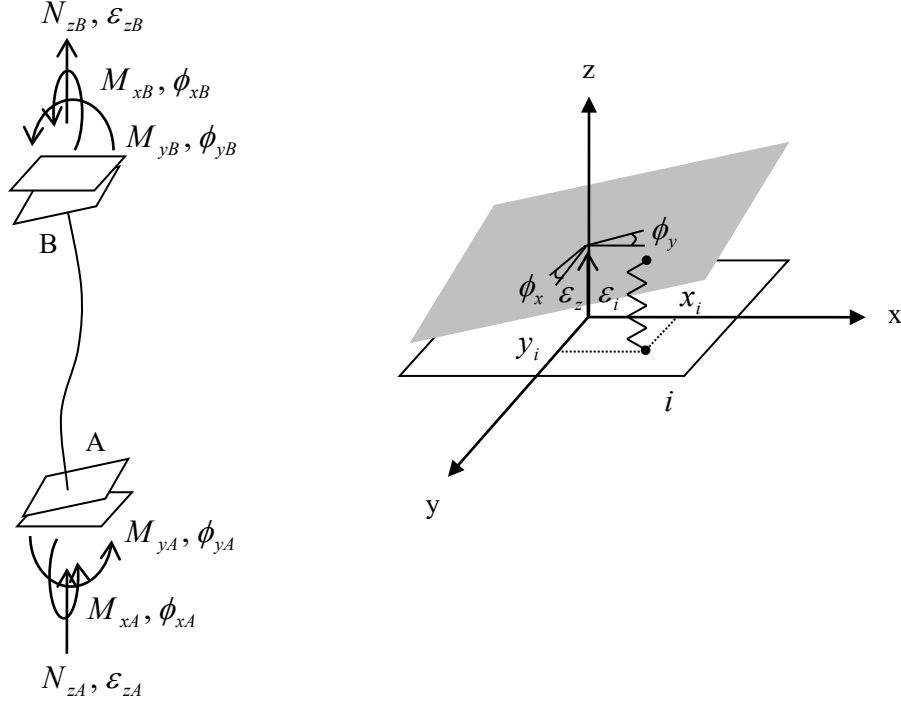


Figure 3-2-5 Nonlinear bending springs

Displacement of the  $i$ -th nonlinear axial spring is,

$$\epsilon_i = \epsilon_z - y_i \phi_x + x_i \phi_y \quad (3-2-19)$$

Equilibrium condition in the nonlinear section is,

$$\begin{aligned} M'_y &= \sum_i k_i \epsilon_i x_i = \sum_i k_i (\epsilon_z - y_i \phi_x + x_i \phi_y) x_i \\ M'_x &= -\sum_i k_i \epsilon_i y_i = -\sum_i k_i (\epsilon_z - y_i \phi_x + x_i \phi_y) y_i \\ N'_z &= \sum_i k_i \epsilon_i = \sum_i k_i (\epsilon_z - y_i \phi_x + x_i \phi_y) \end{aligned} \quad (3-2-20)$$

In a matrix form

$$\begin{Bmatrix} M'_y \\ M'_x \\ N'_z \end{Bmatrix} = \begin{bmatrix} \sum_i k_i x_i^2 & -\sum_i k_i x_i y_i & \sum_i k_i x_i \\ & \sum_i k_i y_i^2 & -\sum_i k_i y_i \\ sym. & & \sum_i k_i \end{bmatrix} \begin{Bmatrix} \phi_y \\ \phi_x \\ \epsilon_z \end{Bmatrix} = [k_p] \begin{Bmatrix} \phi_y \\ \phi_x \\ \epsilon_z \end{Bmatrix} \quad (3-2-21)$$

Therefore

$$\begin{Bmatrix} \phi_y \\ \phi_x \\ \varepsilon_z \end{Bmatrix} = [k_p]^{-1} \begin{Bmatrix} M'_y \\ M'_x \\ N'_z \end{Bmatrix} = [f_p] \begin{Bmatrix} M'_y \\ M'_x \\ N'_z \end{Bmatrix} \quad (3-2-22)$$

For both ends

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \\ \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \\ M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} \quad (3-2-23)$$

### c) Nonlinear vertical springs

The nonlinear bending spring is constructed from the nonlinear vertical springs arranged in the member section as shown in Figure 3-2-5. This model is called “Multi-spring model” proposed by S. S. Lai, G. T. Will and S. Otani (1984) and modified by K-N. Li (1988). The section is divided in 5 areas; where 4 corner areas have steel springs and concrete springs and the center area has one concrete spring.

The strength and the location of nonlinear springs are obtained from the equilibrium condition under the balance axial force,  $N_b$ .

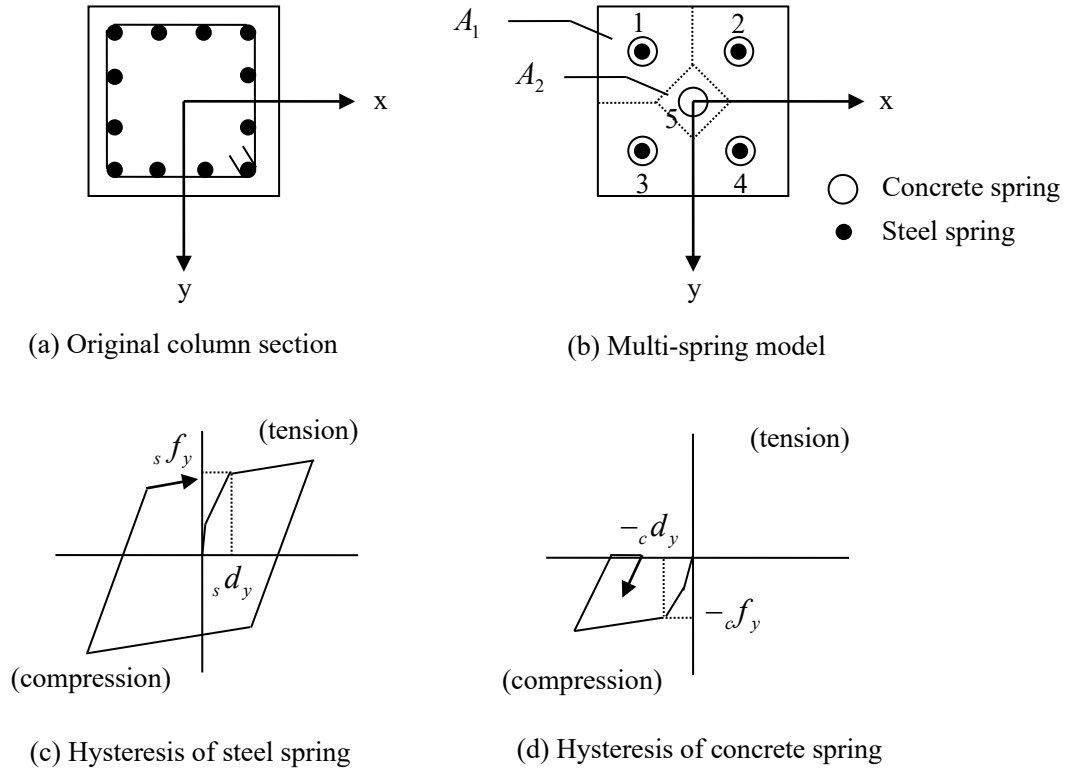


Figure 3-2-6 Nonlinear vertical springs

#### Strength of steel spring

The strength of the steel spring is one-fourth of total strength of rebars in the section, i.e.,

$${}_s f_y = \frac{A_s \sigma_y}{4} \quad (3-2-24)$$

where,

- $A_s$  : Total area of rebar in the section
- $\sigma_y$  : Strength of rebar

### Strength of concrete spring

As shown in Figure 3-2-6, the strength of the corner concrete spring is obtained from the equilibrium condition in the vertical direction under the balance axial force,  $N_b \approx -0.4bD\sigma_B$ , that is,

$${}_c f_{y1} = \frac{N_b}{2} = 0.2bD\sigma_B \quad (3-2-25)$$

Therefore, the area of the corner concrete,  $A_1$ , is,

$$A_1 = \frac{{}_c f_y}{(0.85\sigma_B)} \quad (3-2-26)$$

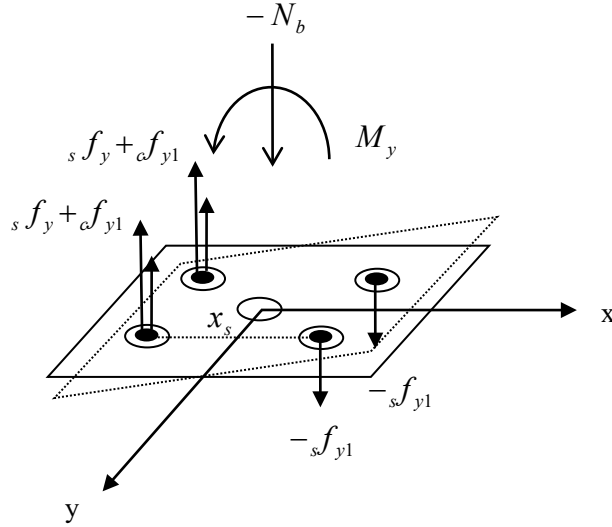


Figure 3-2-7 Equilibrium condition in the column section

The area of the center concrete,  $A_2$ , is the rest of the area of the section,

$$A_2 = bD - 4A_1 \quad (> 0) \quad (3-2-27)$$

The strength of the center concrete spring is then obtained as,

$${}_c f_{y2} = 0.85k\sigma_B A_2 \quad (3-2-28)$$

where,  $k$  is the confined effect ( $k = 1.3$ ) of the concrete.

### Location of vertical springs

The distance between the corner springs,  $x_s$ , is obtained from the equilibrium condition regarding the moment force in Figure 3-2-7,

$$M_y = x_s (2{}_s f_y + c f_{y1}) = x_s (2{}_s f_y + 0.5N_b) \quad (3-2-29)$$

Therefore,

$$x_s = \frac{M_y}{2{}_s f_y + 0.5N_b} \quad (3-2-30)$$

Note that  $M_y$  is calculated from Equation (3-2-6) for the balance axial force,  $N = N_b$ .

### Example)

To verify the efficiency of the Multi-Spring model for the column element, the M-N relationship is compared between MS-model and theory using one column element. The column section is shown in the Figure below:

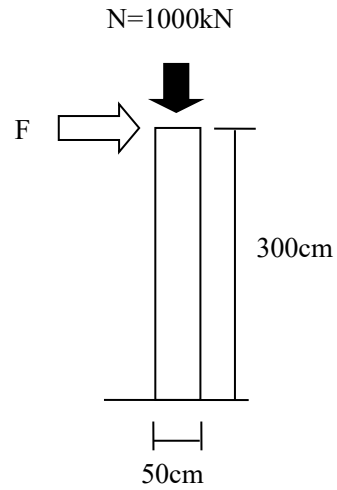
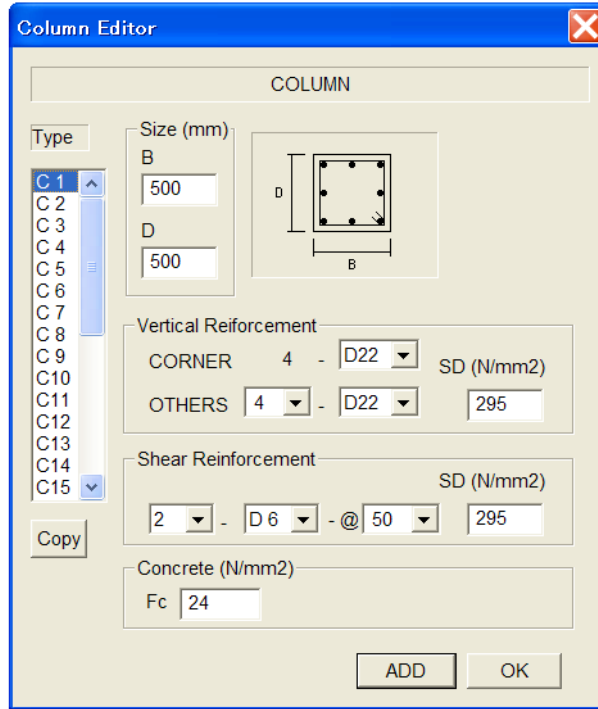


Figure 3-2-8

Firstly, the strengths and locations of vertical springs are calculated as

$$a_t = 15.484 (cm^2) \quad \sigma_y = 1.1f_y = 32.45 (kN / cm^2) \quad \sigma_B = 2.4 (kN / cm^2)$$

$$N_b = 0.4bD\sigma_B = 2400 (kN) \quad N_{max} = bD\sigma_B + A_s\sigma_y = 6502 (kN)$$

$${}_s f_y = 251.2 (kN) \quad {}_s f_{y1} = 1200 (kN) \quad {}_s f_{y2} = 390 (kN) \quad x_s = 30 (cm)$$

In the range  $(0 < N \leq N_b)$ , the Multi-Spring model gives

$$M_y = (2{}_s f_y + 0.5N)x_s$$

which is plotted as the solid line in Figure 3-2-8. The results of Multi-Spring model give smaller values than theoretical results in the range  $0 < N < Nb$ .

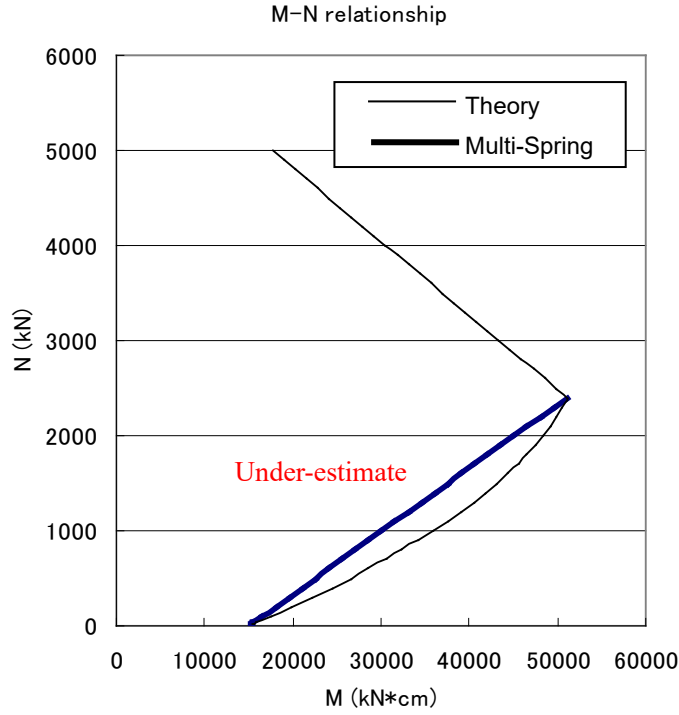


Figure 3-2-9 Comparison of M-N relationship

K-N. Li (1988) proposed to use the following formulation for deciding the location of vertical springs instead of Equation (3-2-29), as follows:

$$x_s = \frac{M_{y0}}{2_s f_y + 0.5 N_0} \quad (3-2-31)$$

where,  $N_0$  is the axial force from the dead loads and the live loads acting on the column ( $N_0 < N_b$ ), and  $M_{y0}$  is the yield moment under the axial force  $N_0$ , that is:

$$M_{y0} = 0.8 a_t \sigma_y D + 0.5 N_0 D \left( 1 - \frac{N_0}{b D \sigma_B} \right) \quad (3-2-32)$$

For the example column, assuming  $N_0 = 1000$  (kN),

$$x_s = 35.8 \text{ (cm)}$$

The yield moment is plotted as the solid line in Figure 3-2-9. It improves the results of Multi-Spring model.

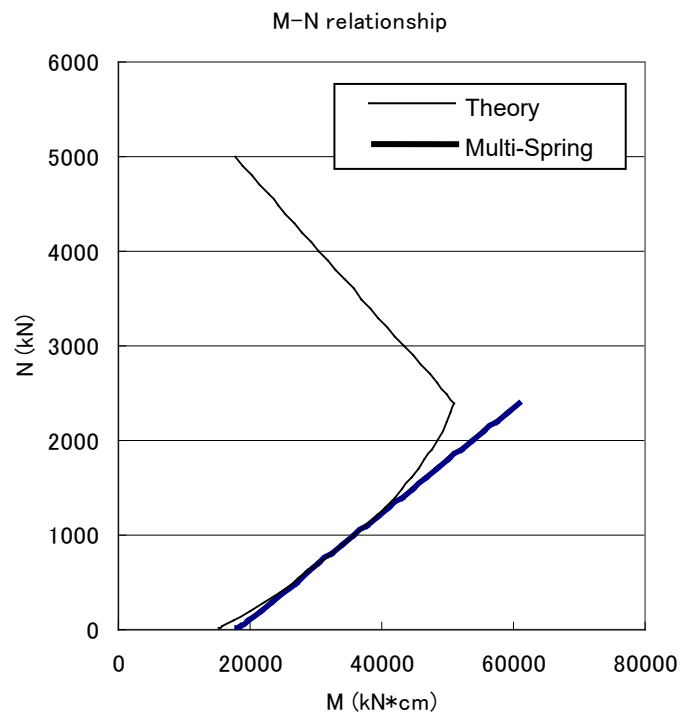


Figure 3-2-10 Comparison of M-N relationship

### Yield displacement of vertical spring

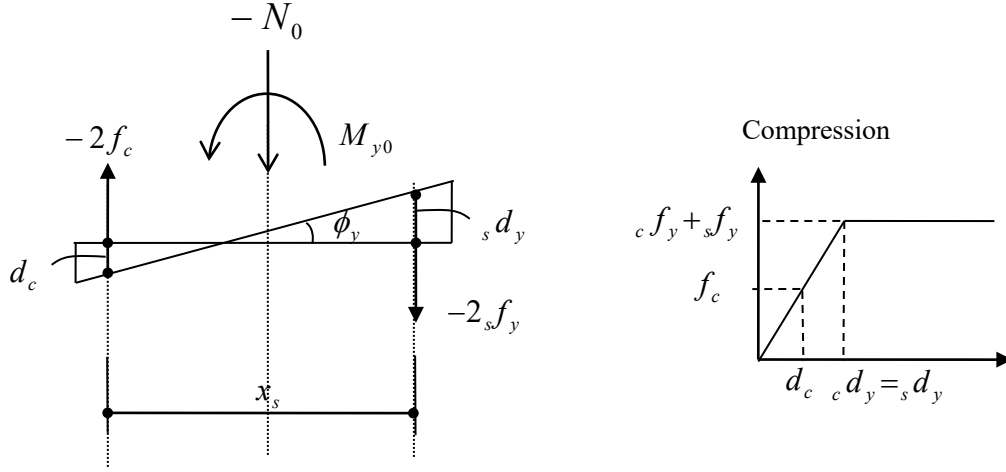


Figure 3-2-11 Equilibrium condition under the axial force  $N_0$

From the equilibrium condition under the axial force  $N_0$  as shown in the above Figure, the yield displacement of the tension side steel spring,  $_s d_y$ , is obtained as follows:

$$\begin{aligned}
 _s d_y + d_c &= \phi_y x_s \\
 d_c &= \frac{f_c}{_s f_y + _c f_y} _s d_y \\
 f_c &= \frac{N_0 + 2 _s f_y}{2} \\
 _s d_y &= \frac{\phi_y x_s}{1 + \frac{N_0 + 2 _s f_y}{2 _s f_y + 2 _c f_y}}
 \end{aligned} \tag{3-2-33}$$

The yield displacement of concrete spring,  $_c d_y$ , is assumed to be the same as that of the steel spring,

$$_c d_y = _s d_y \tag{3-2-34}$$

#### d) Nonlinear shear spring

##### d-1) Force-deformation relationship

There are two nonlinear shear springs in x and y directions. Hysteresis model of the nonlinear shear springs is the same as that in the beam element.

##### Yield shear force

The yield shear force,  $Q_y$  is calculated as,

$$Q_y = \left\{ \frac{0.053 p_t^{0.23} (\sigma_B + 18)}{M/(QD) + 0.12} + 0.85 \sqrt{p_w \cdot \sigma_{wy}} + 0.1 \sigma_0 \right\} b \cdot j \quad (3-2-35)$$

where,

$p_t$	:	Tensile reinforcement ratio
$\sigma_B$	:	Compression strength of concrete
$M/(QD)$	:	$\approx$ Shear span-to-depth ratio ( $= l/(2D)$ )
$p_w$	:	Shear reinforcement ratio
$\sigma_{wy}$	:	Strength of shear reinforcement
$\sigma_0$	:	Axial stress of the column
$j$	:	Distance between the centers of stress in the section ( $= (7/8)d$ ).

##### Crack shear force

The crack shear force is,  $Q_c$ , is assumed as,

$$Q_c = 0.3 Q_y \quad (3-2-36)$$

##### Ultimate shear force

The ultimate shear force is,  $Q_u$ , is assumed as,

$$Q_u = Q_y + k_{y3} (s_u - s_y) \quad (3-2-37)$$

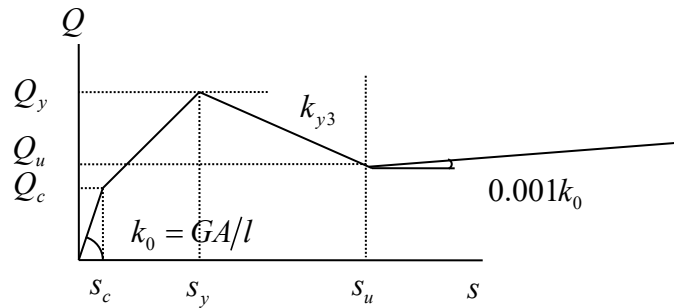


Figure 3-2-12 Shear force - deformation relationship

### Crack shear deformation

The crack shear deformation is obtained as,

$$s_c = \gamma_c l, \quad \gamma_c = \frac{Q_c}{GA} \quad (3-2-38)$$

### Yield shear displacement

The yield shear deformation is assumed as,

$$s_y = \gamma_y l, \quad \gamma_y = \frac{1}{250} \quad (3-2-39)$$

### Ultimate shear displacement

The ultimate shear deformation is assumed as,

$$s_u = \gamma_u l, \quad \gamma_u = \frac{1}{100} \quad (3-2-40)$$

The poly-linear slip model (see Appendix) is adopted for the hysteresis of the shear spring.

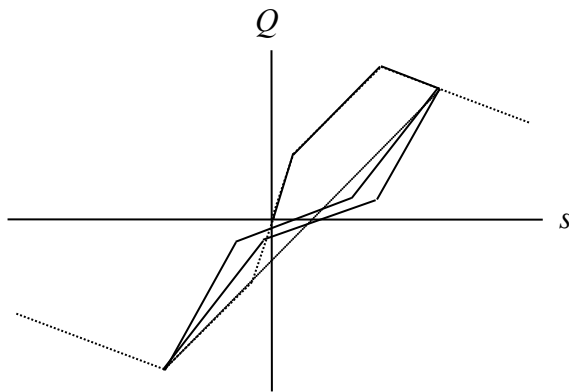


Figure 3-2-13 Poly-linear slip model for shear spring

The parameters on the backbone curve can be changed in the Option Menu of Column element. The default values are given as follows:

Nonlinear Shear Spring

Qc = 0.3

Qy

(K0 = GA)

Ry = 0.004

: Yield shear angle

Ru = 0.01

: Ultimate shear angle

141

Example)

Column Editor

COLUMN

Type

C1  
C2  
C3  
C4  
C5  
C6  
C7  
C8  
C9  
C10  
C11  
C12  
C13  
C14  
C15  
C16

Copy

Size (mm)

B 600 d1 40

D 600 d2 40

Y-side

X-side

Main Reinforcement Bar

corner 4 - D22

X-side 2 - D22 (N/mm2)

Y-side 2 - D22 SD 295

Shear Reinforcement Bar

X-side 2 - D13 - @ 100

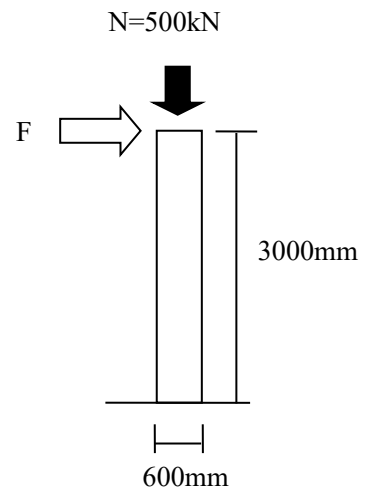
Y-side 2 - D13 - @ 100 SD 295

Concrete (N/mm2)

Fc 24

OPTION

Import Export ADD OK



$$Q_y = \left\{ \frac{0.053 p_t^{0.23} (\sigma_B + 18)}{M / (QD) + 0.12} + 0.85 \sqrt{p_w \cdot \sigma_{wy}} + 0.1 \sigma_0 \right\} b \cdot j \quad (3-2-35)$$

where,

$$b = 600 \text{ (mm)}, \quad j = 0.8 \cdot d = 480 \text{ (mm)}$$

$$p_t = 0.32 \text{ (%)}, \quad \sigma_B = 240 \text{ (N/mm}^2\text{)}, \quad M / (QD) \approx l / (2D) = 3000 / (2 \cdot 600) = 2.5$$

$$p_w = 100 \cdot a_w / (b \cdot x) = 0.0042, \quad a_w = 2 \text{ D13} = 253 \text{ (mm}^2\text{)}, \quad x = 100 \text{ (mm)}$$

$$\sigma_{wy} = 1.1(295) = 324.5 \text{ (N/mm}^2\text{)}, \quad \sigma_0 = 1.388 \text{ (N/mm}^2\text{)}$$

$$Q_y = 479.2 \text{ (kN)}$$

## d-2) Shear spring model 1

**Case 1: In the case that shear springs in x and y directions are independently defined**

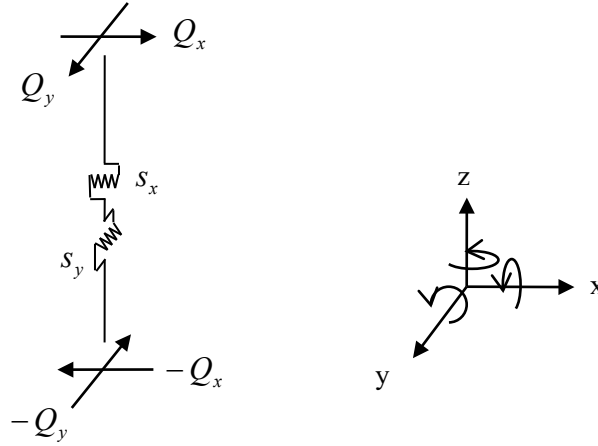


Figure 3-2-14 Nonlinear shear springs in column

The force-deformation relationship of shear spring is

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} k_{sx} & 0 \\ 0 & k_{sy} \end{bmatrix} \begin{Bmatrix} s_x \\ s_y \end{Bmatrix} \quad (3-2-41)$$

### d-3) Shear spring model 2

**Case 2: In the case that nonlinear interaction between shear and axial components is considered**

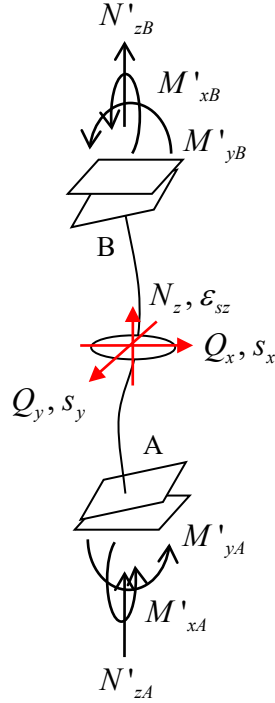


Figure 3-2-15 Nonlinear shear springs

The force-deformation relationship of shear spring is

$$\begin{Bmatrix} Q_x \\ Q_y \\ N_z \end{Bmatrix} = [k_{sp}] \begin{Bmatrix} s_x \\ s_y \\ \epsilon_{sz} \end{Bmatrix} \quad (3-2-42)$$

The stiffness matrix  $[k_{sp}]$  is obtained by the Plastic Theory as explained in the **Appendix (not implemented)**.

### e) Modification of initial stiffness of nonlinear springs

The same modification can be done for the nonlinear springs of column element as described for those of beam element by reducing the initial stiffness of the nonlinear spring and increasing the stiffness of the elastic element as shown in the following figure:

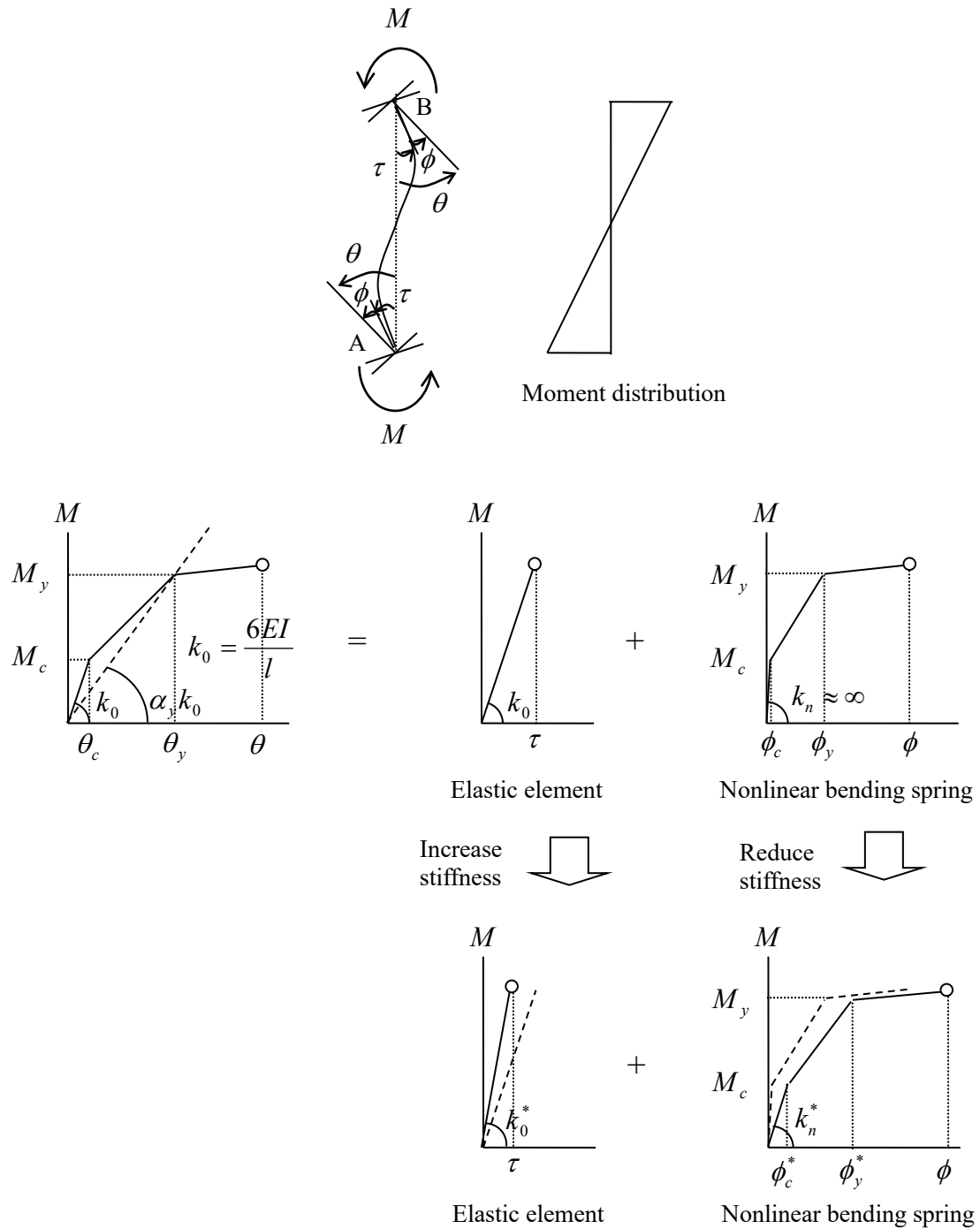


Figure 3-2-16 Modification of moment – rotation relationship

Introducing the concept of “plastic zones”, the initial stiffness of the i-th multi-spring can be expressed as,

$$k_0^i = \frac{E_i A_i}{p_z} \quad (3-2-43)$$

where  $E_i$  : the material young's modulus,  $A_i$  : the spring governed area, and  $p_z$  : the length of assumed plastic zone. When  $p_z \rightarrow 0$ , it represents the infinite stiffness for rigid condition.

From Equation (3-2-20), when we consider the flexural flexibility in x-z plane, the flexibility matrix for the nonlinear MS section is,

$$\begin{Bmatrix} \phi_y \\ \varepsilon_z \end{Bmatrix} = \begin{bmatrix} 1/\sum_i k_0^i x_i^2 & 0 \\ 0 & 1/\sum_i k_0^i \end{bmatrix} \begin{Bmatrix} M'_y \\ N'_z \end{Bmatrix} = \begin{bmatrix} p_z/\sum_i E_i A_i x_i^2 & 0 \\ 0 & p_z/\sum_i E_i A \end{bmatrix} \begin{Bmatrix} M'_y \\ N'_z \end{Bmatrix} \quad (3-2-44)$$

Also, introducing the flexibility reduction factors,  $\gamma_0 (< 0)$ ,  $\gamma_1 (< 0)$ ,  $\gamma_2 (< 0)$ , the flexibility matrix of the elastic element is,

$$[f_c] = \begin{bmatrix} \gamma_1 \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} \\ -\frac{l'}{6EI_y} & \gamma_2 \frac{l'}{3EI_y} \\ & & \gamma_0 \frac{l'}{EA} \end{bmatrix} \quad (3-2-45)$$

Making the modified flexibility matrix to be identical to the original one,

$$\begin{bmatrix} \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ & \frac{l'}{3EI_y} & 0 \\ sym. & & \frac{l'}{EA} \end{bmatrix}_{original} = \begin{bmatrix} \frac{p_{z1}}{\sum_i E_i A_i x_i^2} + \gamma_1 \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ & \frac{p_{z2}}{\sum_i E_i A_i x_i^2} + \gamma_2 \frac{l'}{3EI_y} & 0 \\ sym. & & \frac{p_{z1}}{\sum_i E_i A} + \frac{p_{z2}}{\sum_i E_i A} + \gamma_0 \frac{l'}{EA} \end{bmatrix}_{modified} \quad (3-2-46)$$

Since  $\sum_i A_i x_i^2 \approx I_y$ , this gives the flexivility reduction factors as:

$$\gamma_1 = 1 - \frac{3}{l'} p_{z1}, \quad \gamma_2 = 1 - \frac{3}{l'} p_{z2}, \quad \gamma_0 = 1 - \frac{1}{l'} (p_{z1} + p_{z2}) \quad (3-2-47)$$

Adopting  $p_{z1} = p_{z2} = \frac{l'}{10}$  as discussed for beam element, the reduction factors will be:

$$\gamma_1 = \gamma_2 = 0.7, \quad \gamma_0 = 0.8 \quad (3-2-48)$$

## References

- 1) S. S. Lai, G. T. Will, and S. Otani (1984), “Model for Inelastic Biaxial Bending of Concrete Members,” *Journal of Structural Division, ASCE*, Vol. 110, ST1, 1984, pp.2563-2584.
- 2) K-N. Li (1988), “Nonlinear Earthquake Response of Reinforced Concrete Space Frames,” the dissertation for the degree of Doctor in University of Tokyo (in Japanese), 1988.12.
- 3) K-N. Li (2004), CANNY, Technical Manual.

### 3.2.2 Steel Column

#### a) Section properties

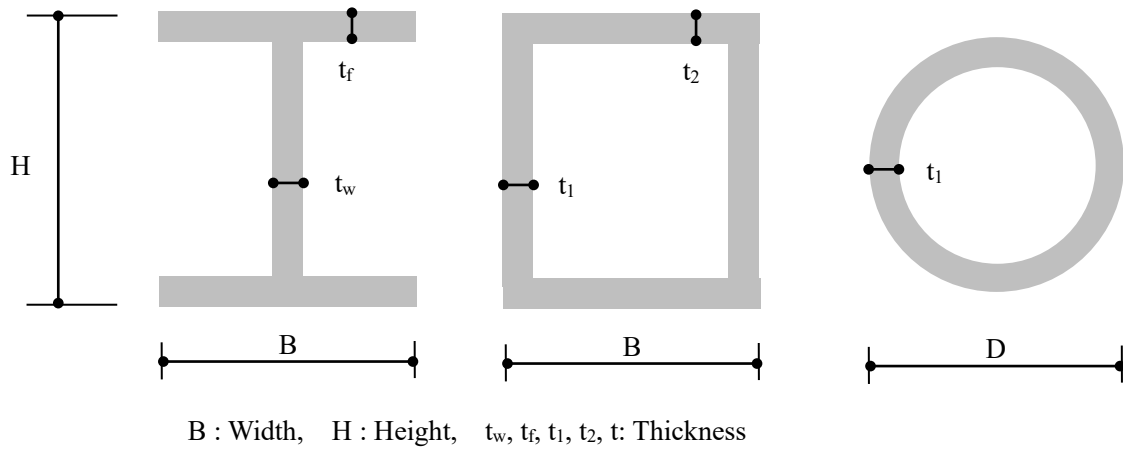


Figure 3-2-17 Steel Column Section

Area of section to calculate axial deformation

$$A_N = \text{total area of section} \quad (3-2-49)$$

Area of section to calculate shear deformation

$$A_S = ( \text{shaded area} ) \quad (3-2-50)$$



Figure 3-2-18 Area of section for shear

Moment of inertia around the center of the section

1) H section

$$I = \frac{BH^3 - (B - t_w)(H - 2t_f)^3}{12} \quad : \text{along strong axis} \quad (3-2-51)$$

$$I = \frac{2t_f B^3 + (H - 2t_f)t_w^3}{12} \quad : \text{along weak axis} \quad (3-2-52)$$

2) Box section

$$I = \frac{BH^3 - (B - 2t_1)(H - 2t_2)^3}{12} \quad (3-2-53)$$

3) Circle section

$$I = \frac{\pi}{64} [D^4 - (D - 2t)^4] \quad (3-2-54)$$

Moment of inertia for torsion

1) H section

$$J = \frac{2Bt_f^3 + (H - 2t_f)t_w^3}{3} \quad (3-2-55)$$

4) Box section

$$J = \frac{2t_1t_2(B - t_1)^2(H - t_2)^2}{BHt_1t_2 - t_1^2 - t_2^2} \quad (3-2-56)$$

5) Circle section

$$J = \frac{\pi}{32} [D^4 - (D - t)^4] \quad (3-2-57)$$

## b) Nonlinear bending spring

To consider nonlinear interaction among  $M_x - M_y - N_z$ , the nonlinear bending spring at the member end is constructed from the nonlinear vertical springs arranged in the member section as shown in the following Figure.

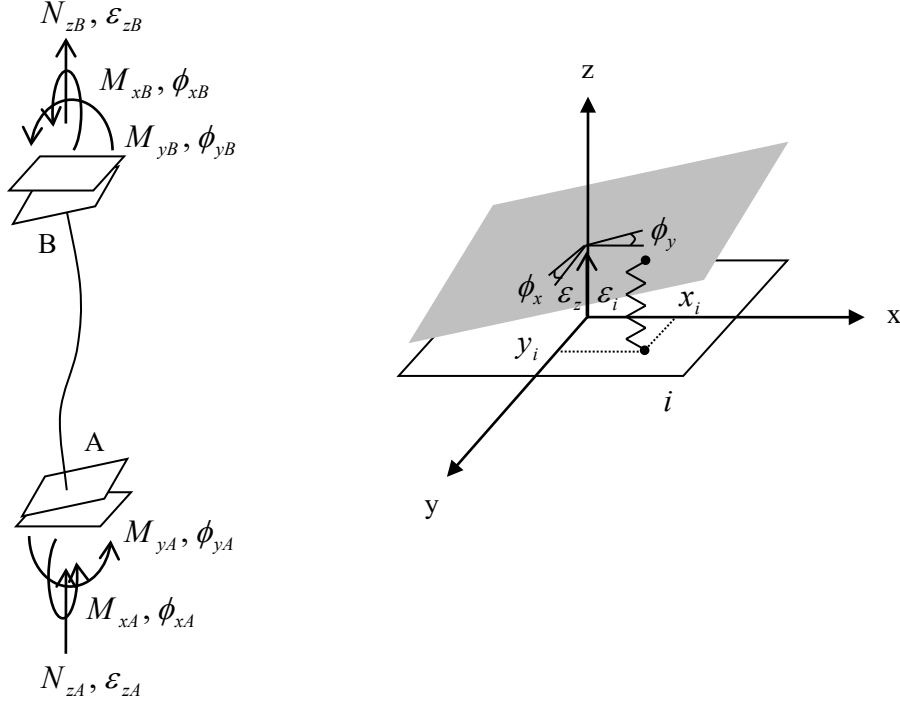


Figure 3-2-19 Nonlinear bending springs

Displacement of the  $i$ -th nonlinear axial spring is,

$$\varepsilon_i = \varepsilon_z - y_i \phi_x + x_i \phi_y \quad (3-2-58)$$

Equilibrium condition in the nonlinear section is,

$$\begin{aligned} M'_y &= \sum_i k_i \varepsilon_i x_i = \sum_i k_i (\varepsilon_z - y_i \phi_x + x_i \phi_y) x_i \\ M'_x &= -\sum_i k_i \varepsilon_i y_i = -\sum_i k_i (\varepsilon_z - y_i \phi_x + x_i \phi_y) y_i \\ N'_z &= \sum_i k_i \varepsilon_i = \sum_i k_i (\varepsilon_z - y_i \phi_x + x_i \phi_y) \end{aligned} \quad (3-2-59)$$

In a matrix form

$$\begin{Bmatrix} M'_y \\ M'_x \\ N'_z \end{Bmatrix} = \begin{bmatrix} \sum_i k_i x_i^2 & -\sum_i k_i x_i y_i & \sum_i k_i x_i \\ & \sum_i k_i y_i^2 & -\sum_i k_i y_i \\ sym. & & \sum_i k_i \end{bmatrix} \begin{Bmatrix} \phi_y \\ \phi_x \\ \varepsilon_z \end{Bmatrix} = [k_p] \begin{Bmatrix} \phi_y \\ \phi_x \\ \varepsilon_z \end{Bmatrix} \quad (3-2-60)$$

Therefore

$$\begin{Bmatrix} \phi_y \\ \phi_x \\ \varepsilon_z \end{Bmatrix} = [k_p]^{-1} \begin{Bmatrix} M'_y \\ M'_x \\ N'_z \end{Bmatrix} = [f_p] \begin{Bmatrix} M'_y \\ M'_x \\ N'_z \end{Bmatrix} \quad (3-2-61)$$

For both ends

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \\ \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \\ M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} \quad (3-2-62)$$

Hysteresis model of nonlinear bending spring is defined as the moment-rotation relationship under the anti-symmetry loading as shown in Figure 3-2-20. The initial stiffness of the nonlinear spring is supposed to be infinite, however, in numerical calculation, a large enough value is used for the stiffness.

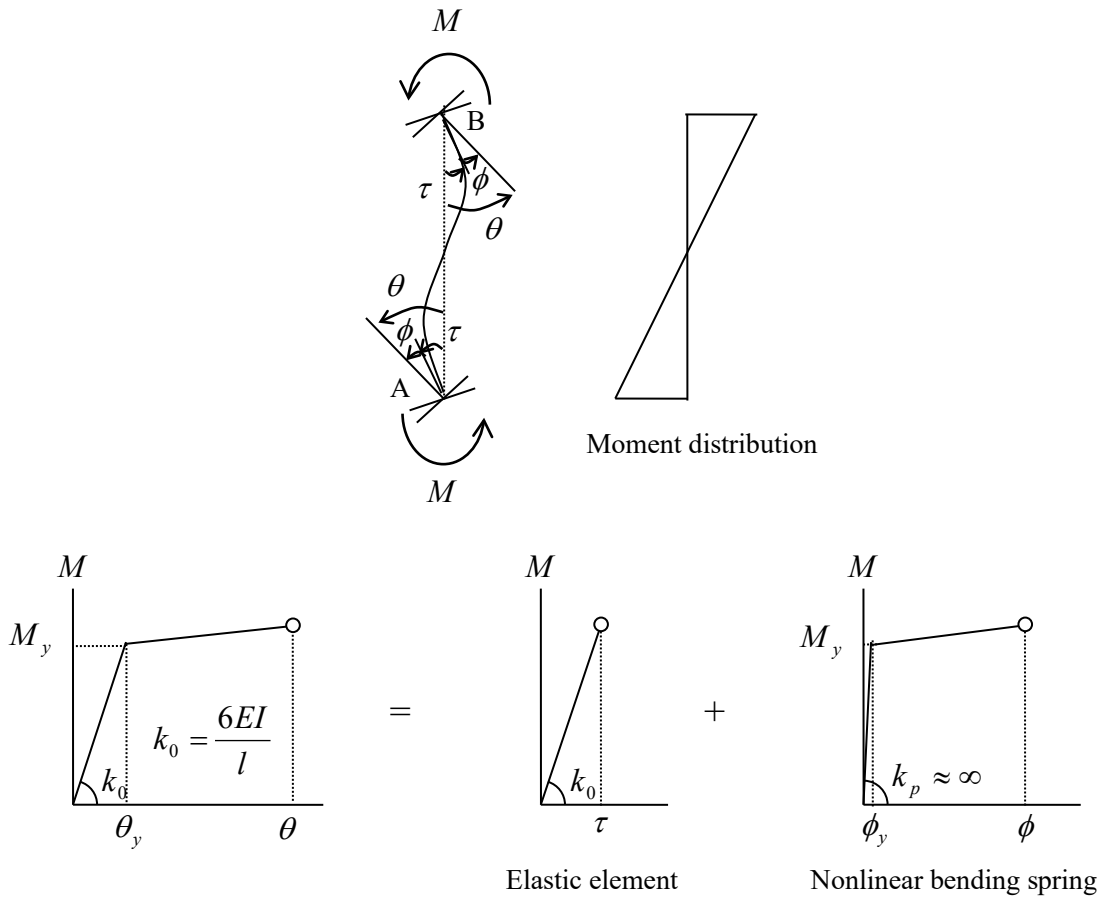
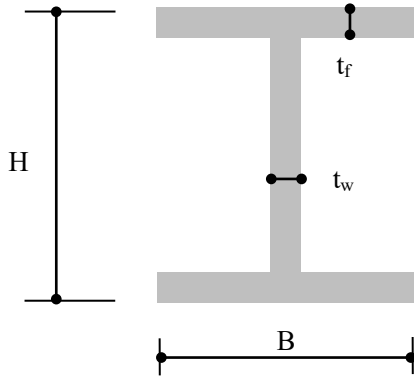


Figure 3-2-20 Moment – rotation relationship at bending spring

## Yield moment force (full plastic moment)

### 1) I shape



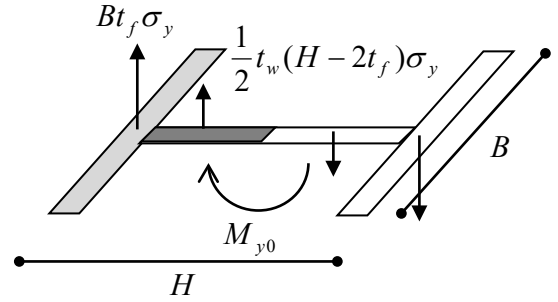
- a) When the neutral axis is inside the web, i.e.,  $N < A_w \sigma_y = t_w (H - 2t_f) \sigma_y$

$$M_y = M_{y0} - y_0^2 t_w \sigma_y \quad (3-2-63)$$

where

$$M_{y0} = \left[ B t_f (H - t_f) + \frac{1}{4} t_w (H - 2t_f)^2 \right] \sigma_y$$

$$y_0 = \frac{N}{2 t_w \sigma_y}$$



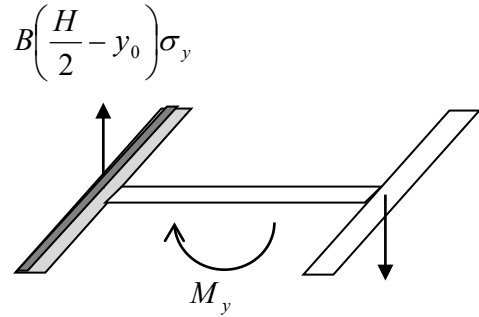
- b) When the neutral axis is inside the flange, i.e.,  $N > A_w \sigma_y = t_w (H - 2t_f) \sigma_y$

$$M_y = B \left( \frac{H}{2} - y_0 \right) \left( \frac{H}{2} + y_0 \right) \sigma_y \quad (3-2-64)$$

where

$$y_0 = \frac{1}{2} \left( \frac{N - N_y}{B \sigma_y} + H \right)$$

$$N_y = [2 B t_f + t_w (H - 2t_f)] \sigma_y$$



## 2) H shape

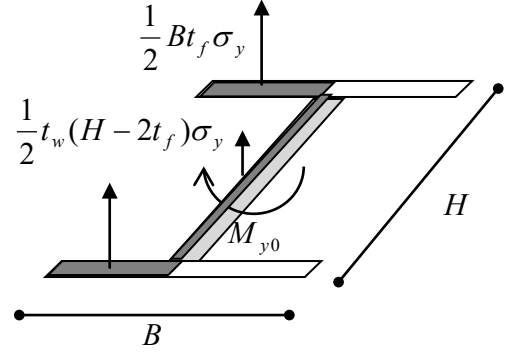
- a) When the neutral axis is inside the web, i.e.,  $N < A_w \sigma_y = t_w H \sigma_y$

$$M_y = M_{y0} - y_0^2 H \sigma_y \quad (3-2-65)$$

where

$$M_{y0} = \left[ \frac{1}{2} B^2 t_f + \frac{1}{4} t_w^2 (H - 2t_f) \right] \sigma_y$$

$$y_0 = \frac{N}{2H\sigma_y}$$



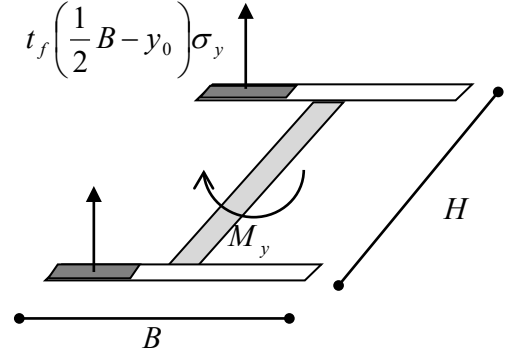
- b) When the neutral axis is inside the web, i.e.,  $N < A_w \sigma_y = t_w H \sigma_y$

$$M_y = 2t_f \left( \frac{B}{2} - y_0 \right) \left( \frac{B}{2} + y_0 \right) \sigma_y \quad (3-2-66)$$

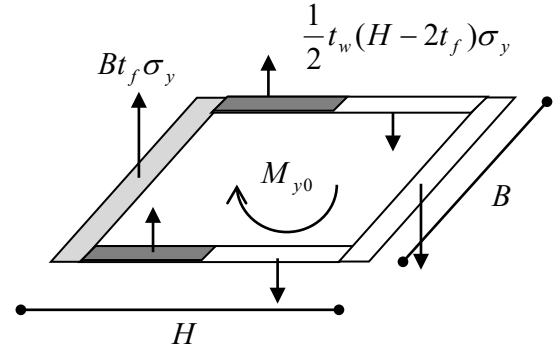
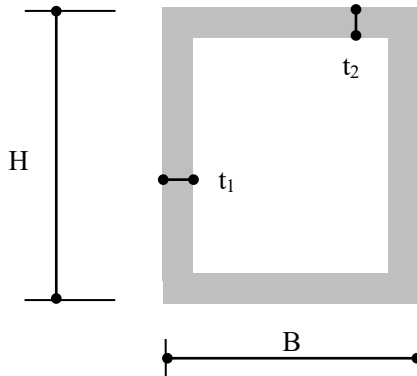
where

$$y_0 = \frac{1}{2} \left( \frac{N - N_y}{2t_f \sigma_y} + B \right)$$

$$N_y = [2Bt_f + t_w(H - 2t_f)] \sigma_y$$



## 3) Box shape



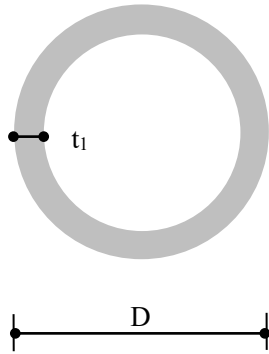
- a) Moment around x-axis

$$M_y = M_y \text{ (I shape by changing } t_w \rightarrow 2t_1, t_f \rightarrow t_2) \quad (3-2-67)$$

- b) Moment around y-axis

$$M_y = M_y \text{ (I shape by changing } t_w \rightarrow 2t_2, t_f \rightarrow t_1, B \leftrightarrow H) \quad (3-2-68)$$

#### 4) Circle shape



$$M_y = M_{y0} \cos\left(\frac{\pi N}{2N_y}\right) \quad (3-2-69)$$

where

$$M_{y0} = (D - t_1)^2 t_1 \sigma_y$$

$$N_y = \pi (D - t_1) t_1 \sigma_y$$

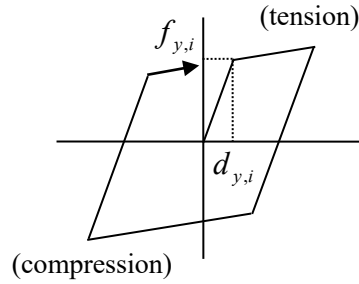
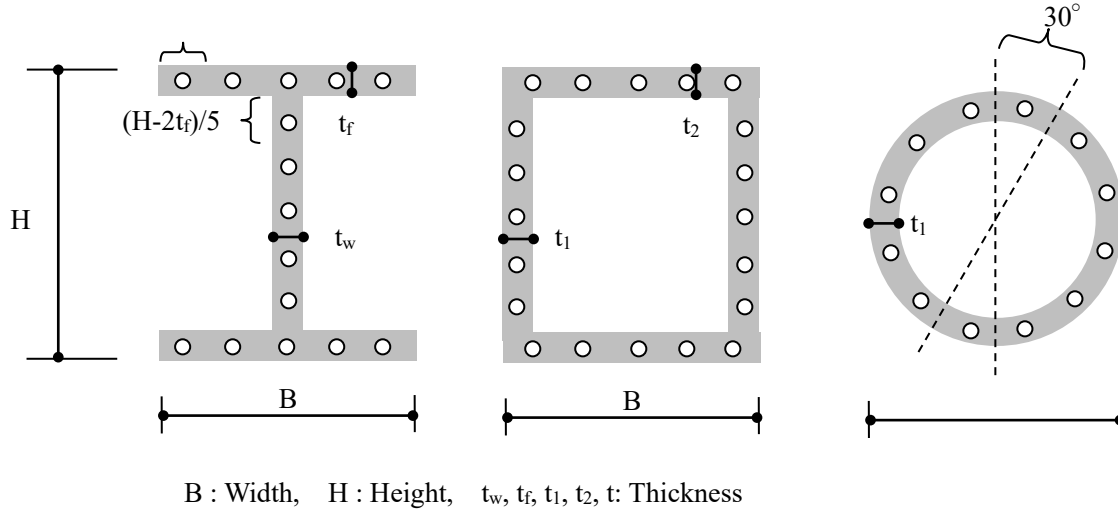
#### Yield rotation

The yield rotation is

$$\theta_y = M_y / k_0, \quad k_0 = \frac{6EI}{l} \quad (3-2-70)$$

### c) Nonlinear vertical springs

The nonlinear bending spring is constructed from the nonlinear vertical springs arranged in the member section as shown in Figure 3-2-21. This model is called “fiber model”. The section is divided in several areas which have steel springs.



Hysteresis of steel spring

Figure 3-2-21 Nonlinear vertical springs

### Strength of steel spring

The strength of the  $i$ -th steel spring is,

$$f_{y,i} = A_i \sigma_y \quad (3-2-71)$$

where,  $A_i$  : the spring governed area,  $\sigma_y$  : the strength of steel

### Yield displacement of steel spring

The yield displacement of the  $i$ -th steel spring is,

$$d_{y,i} = f_{y,i} / k_0^i, \quad k_0^i = E_s A_i \quad (3-2-72)$$

where  $E_s$  : the young's modulus of steel

The same modification can be done for the nonlinear springs of column element as described for those of beam element by reducing the initial stiffness of the nonlinear spring and increasing the stiffness of the elastic element as shown in the following figure:

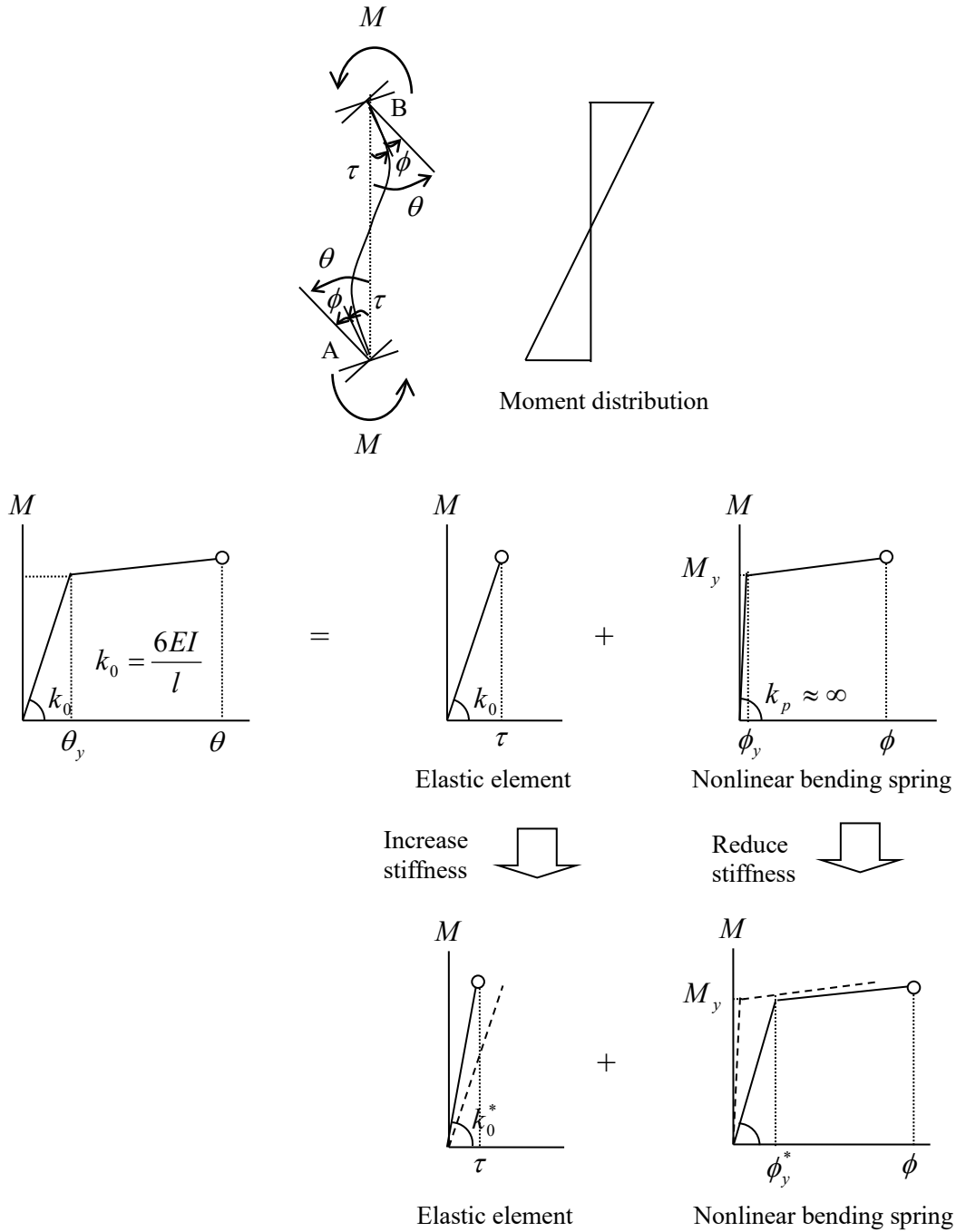


Figure 3-2-22 Modification of moment – rotation relationship

Introducing the concept of “plastic zones”, the initial stiffness of the  $i$ -th multi-spring can be expressed as,

$$k_0^i = \frac{E_i A_i}{p_z} \quad (3-2-73)$$

where  $E_i$  : the material young's modulus,  $A_i$  : the spring governed area, and  $p_z$  : the length of assumed

plastic zone. When  $p_z \rightarrow 0$ , it represents the infinite stiffness for rigid condition.

When we consider the flexural flexibility in x-z plane, the flexibility matrix for the nonlinear MS section is,

$$\begin{Bmatrix} \phi_y \\ \varepsilon_z \end{Bmatrix} = \begin{bmatrix} 1/\sum_i k_0^i x_i^2 & 0 \\ 0 & 1/\sum_i k_0^i \end{bmatrix} \begin{Bmatrix} M'_y \\ N'_z \end{Bmatrix} = \begin{bmatrix} p_z/\sum_i E_i A_i x_i^2 & 0 \\ 0 & p_z/\sum_i E_i A \end{bmatrix} \begin{Bmatrix} M'_y \\ N'_z \end{Bmatrix} \quad (3-2-74)$$

Also, introducing the flexibility reduction factors,  $\gamma_0 (< 0)$ ,  $\gamma_1 (< 0)$ ,  $\gamma_2 (< 0)$ , the flexibility matrix of the elastic element is,

$$[f_C] = \begin{bmatrix} \gamma_1 \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} \\ -\frac{l'}{6EI_y} & \gamma_2 \frac{l'}{3EI_y} \\ & & \gamma_0 \frac{l'}{EA} \end{bmatrix} \quad (3-2-75)$$

Making the modified flexibility matrix to be identical to the original one,

$$\begin{bmatrix} \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ & \frac{l'}{3EI_y} & 0 \\ sym. & & \frac{l'}{EA} \end{bmatrix}_{original} = \begin{bmatrix} \frac{p_{z1}}{\sum_i E_i A_i x_i^2} + \gamma_1 \frac{l'}{3EI_y} & -\frac{l'}{6EI_y} & 0 \\ & \frac{p_{z2}}{\sum_i E_i A_i x_i^2} + \gamma_2 \frac{l'}{3EI_y} & 0 \\ sym. & & \frac{p_{z1}}{\sum_i E_i A} + \frac{p_{z2}}{\sum_i E_i A} + \gamma_0 \frac{l'}{EA} \end{bmatrix}_{modified} \quad (3-2-76)$$

This gives the flexivility reduction factors as:

$$\gamma_1 = 1 - \frac{3}{l'} p_{z1}, \quad \gamma_2 = 1 - \frac{3}{l'} p_{z2}, \quad \gamma_0 = 1 - \frac{1}{l'} (p_{z1} + p_{z2}) \quad (3-2-77)$$

Adopting  $p_{z1} = p_{z2} = \frac{l'}{10}$  as discussed for beam element, the reduction factors will be:

$$\gamma_1 = \gamma_2 = 0.7, \quad \gamma_0 = 0.8 \quad (3-2-78)$$

### 3.2.2 Column with direct input

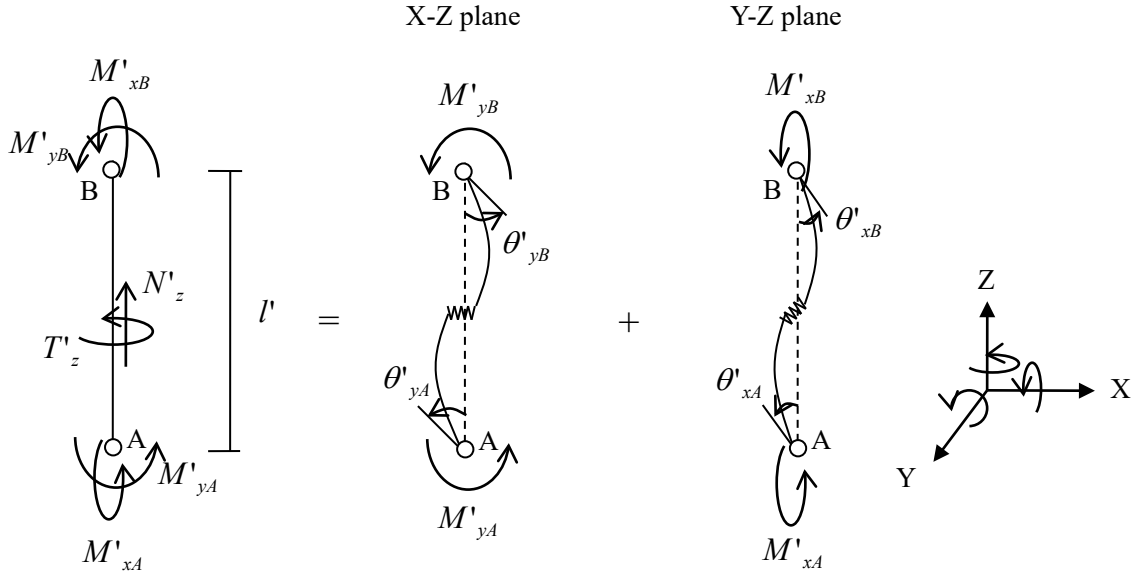


Figure 3-2-23 Element model for column

In case of direct input for Moment-Rotation relationship, we neglect nonlinear interaction among  $M_x - M_y - N_z$  and define the flexural stiffness of nonlinear bending springs in X and Y directions independently. The rotational displacement vector of the nonlinear bending springs will be

$$\begin{Bmatrix} \phi_{yA} \\ \phi_{xA} \\ \varepsilon_{zA} \\ \phi_{yB} \\ \phi_{xB} \\ \varepsilon_{zB} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \\ M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} = \begin{bmatrix} f_{yA} & & & & & \\ & f_{xA} & & & & \\ & & 0 & & & \\ & & & f_{yB} & & \\ & & & & f_{xB} & \\ & & & & & 0 \end{bmatrix} \begin{Bmatrix} M'_{yA} \\ M'_{xA} \\ N'_{zA} \\ M'_{yB} \\ M'_{xB} \\ N'_{zB} \end{Bmatrix} \quad (3-2-79)$$

The displacement vector of the column element is obtained as the sum of the displacement vectors of elastic element, nonlinear shear springs and nonlinear bending springs,

$$\begin{Bmatrix} \theta'_{yA} \\ \theta'_{yB} \\ \theta'_{xA} \\ \theta'_{xB} \\ \delta'_z \\ \theta'_z \end{Bmatrix} = \begin{Bmatrix} \tau'_{yA} \\ \tau'_{yB} \\ \tau'_{xA} \\ \tau'_{xB} \\ \delta''_z \\ \theta'_z \end{Bmatrix}_{\text{elastic element}} + \begin{Bmatrix} \phi_{yA} \\ \phi_{yB} \\ \phi_{xA} \\ \phi_{xB} \\ \varepsilon_z \\ 0 \end{Bmatrix}_{\text{bending spring}} + \begin{Bmatrix} \eta_{yA} \\ \eta_{yB} \\ \eta_{xA} \\ \eta_{xB} \\ 0 \\ 0 \end{Bmatrix}_{\text{shear spring}} = [f_C] \begin{Bmatrix} M'_{yA} \\ M'_{yB} \\ M'_{xA} \\ M'_{xB} \\ N'_z \\ T'_z \end{Bmatrix} \quad (3-2-80)$$

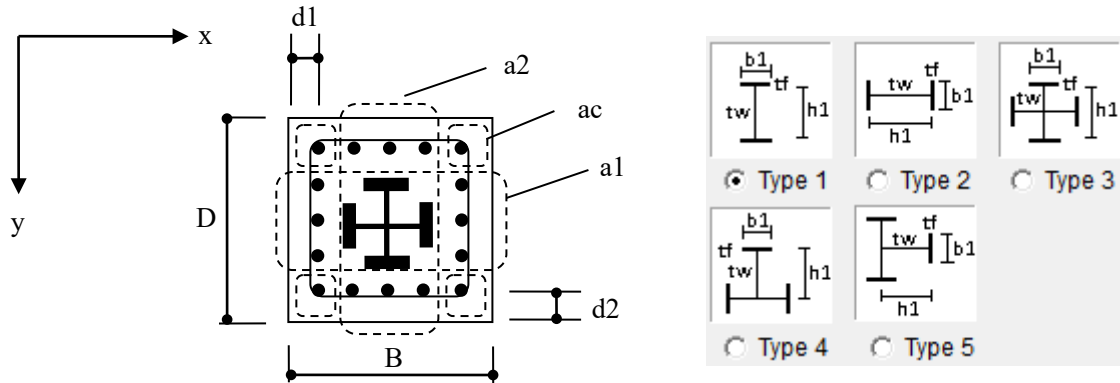
The flexural matrix  $[f_C]$  is;

$$[f_C] = \begin{bmatrix} f_{yA} + \frac{l'}{3EI_y} + \frac{1}{k_{sx}l'^2} & -\frac{l'}{6EI_y} + \frac{1}{k_{sx}l'^2} & & & & \\ & f_{xA} + \frac{l'}{3EI_y} + \frac{1}{k_{sx}l'^2} & & & & 0 \\ & & f_{yB} + \frac{l'}{3EI_x} + \frac{1}{k_{sy}l'^2} & -\frac{l'}{6EI_x} + \frac{1}{k_{sy}l'^2} & & \\ & & & f_{xB} + \frac{l'}{3EI_x} + \frac{1}{k_{sy}l'^2} & & \\ & & & & \frac{l'}{EA} & \\ sym. & & & & & \frac{l'}{GI_z} \end{bmatrix}$$

(3-2-81)

### 3.2.3 SRC Column

#### a) Section properties



- $B$  : Width of beam,
- $D$  : Height of beam,
- $d1$  : Distance to the center of  $x$ -direction main rebars,
- $d2$  : Distance to the center of  $y$ -direction main rebars,
- $a1$  : Area of  $x$ -side main rebars,
- $a2$  : Area of  $y$ -side main rebars,
- $ac$  : Area of corner main rebars
- $b1$  : Width of steel
- $h1$  : Height of steel
- $tw$  : Thickness of web
- $tf$  : Thickness of flange

Figure 3-2-24 RC Column Section

Area of section to calculate axial deformation

$$A_N = BD + (n_E - 1)(a_1 + a_2 + a_c + a_{ST}) \quad (3-2-82)$$

where,

$$n_E = E_s / E_c \quad : \text{Ratio of Young's modulus between steel } (E_s) \text{ and concrete } (E_c)$$

$$a_{ST} = n_f(b_1 - t_w)t_f + n_w h_1 t_w \quad : \text{Area of steel}$$

$$n_f = 2, \quad n_w = 1: \quad \text{Type1, Type2,}$$

$$n_f = 4, \quad n_w = 2: \quad \text{Type3}$$

$$n_f = 3, \quad n_w = 2: \quad \text{Type4, Type5}$$

Area of section to calculate shear deformation

$$A_S = BD / \kappa, \quad \kappa = 1.2 \quad (3-2-83)$$

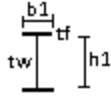
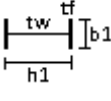
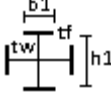
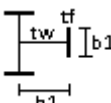
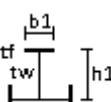
Moment of inertia around the center of the section

$$I_y = \frac{DB^3}{12} + (n_E - 1) \left( (a_c + a_1) \left( \frac{B}{2} - d_1 \right)^2 + I_{S,y} \right) \quad (3-2-84)$$

$$I_x = \frac{BD^3}{12} + (n_E - 1) \left( (a_c + a_2) \left( \frac{D}{2} - d_2 \right)^2 + I_{S,x} \right) \quad (3-2-85)$$

where

$I_S$  : Moment of inertia of steel

	$I_{S,x}$	$I_{S,y}$
Type 1 	$I_I = \frac{1}{12} (b_1 h_1^3 - (b_1 - t_w)(h_1 - 2t_f)^3)$	$I_H = \frac{1}{12} (2t_f b_1^3 + (h_1 - 2t_f)t_w^3)$
Type 2 	$I_H$	$I_I$
Type 3 	$I_I + I_H$	$I_I + I_H$
Type 4 	$I_I + I_H$	$I_I + I_H + A_H \left( \frac{h_1}{2} \right)^2$
Type 5 	$I_I + I_H + A_H \left( \frac{h_1}{2} \right)^2$	$I_I + I_H$

### b) Nonlinear bending spring

Hysteresis model of a nonlinear bending spring is the same as RC beam.

### Crack moment force

For reinforced concrete elements, the crack moment,  $M_c$  is calculated as,

$$M_c = 0.56 \sqrt{\sigma_B} Z_e + \frac{ND}{6} \quad (3-2-86)$$

### Yield moment force

The yield moment,  $M_y$  is calculated as,

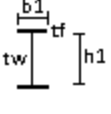
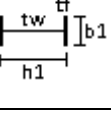
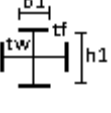
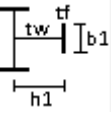
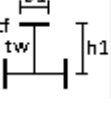
$$M_y = M_{y,RC} + M_{y,S} \quad (3-2-87)$$

where

$M_{y,RC}$  : Yield moment of reinforced concrete

$$M_{y,RC} = 0.8a_t\sigma_y D + 0.5N_b D \left(1 - \frac{N_b}{bD\sigma_B}\right) \quad (3-2-88)$$

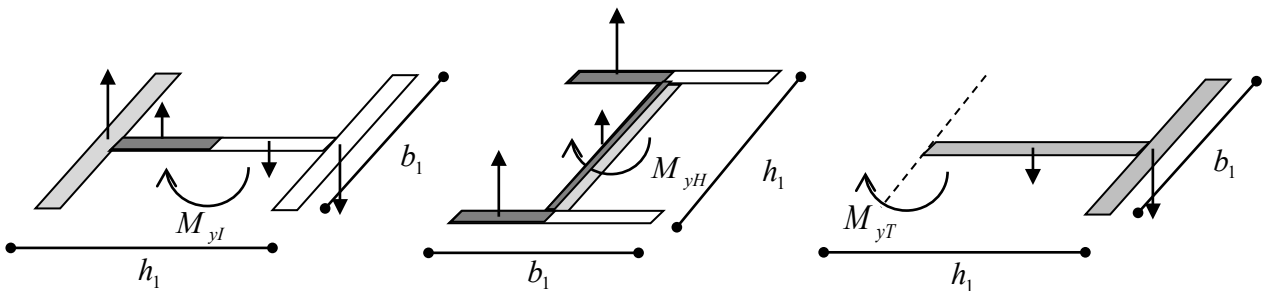
$M_{y,S}$  : Yield moment of steel

	$M_{y,S,x}$	$M_{y,S,y}$
Type 1 	$M_{yI}$	$M_{yH}$
Type 2 	$M_{yH}$	$M_{yI}$
Type 3 	$M_{yI} + M_{yH}$	$M_{yI} + M_{yH}$
Type 4 	$M_{yI} + M_{yH}$	$M_{yI} + M_{yT}$
Type 5 	$M_{yI} + M_{yT}$	$M_{yI} + M_{yH}$

$$M_{yI} = \left[ b_1 t_f (h_1 - t_f) + \frac{1}{4} t_w (h_1 - 2t_f)^2 \right] \sigma_{y,S}$$

$$M_{yH} = \left[ \frac{1}{2} b_1^2 t_f + \frac{1}{4} t_w^2 (h_1 - 2t_f) \right] \sigma_{y,S}$$

$$M_{yT} = \left[ b_1 t_f (h_1 - t_f) + \frac{1}{2} t_w (h_1 - t_f)^2 \right] \sigma_{y,S}$$



## Appendix 3.2:

### A-1. Hysteresis of Steel and Concrete Springs of Multi-Spring Models for RC elements

#### a) Steel spring

For the steel spring, the maximum-oriented model is adopted for the hysteresis before yielding, and the tri-linear model is adopted after yielding as shown in Figure 3-2-15.

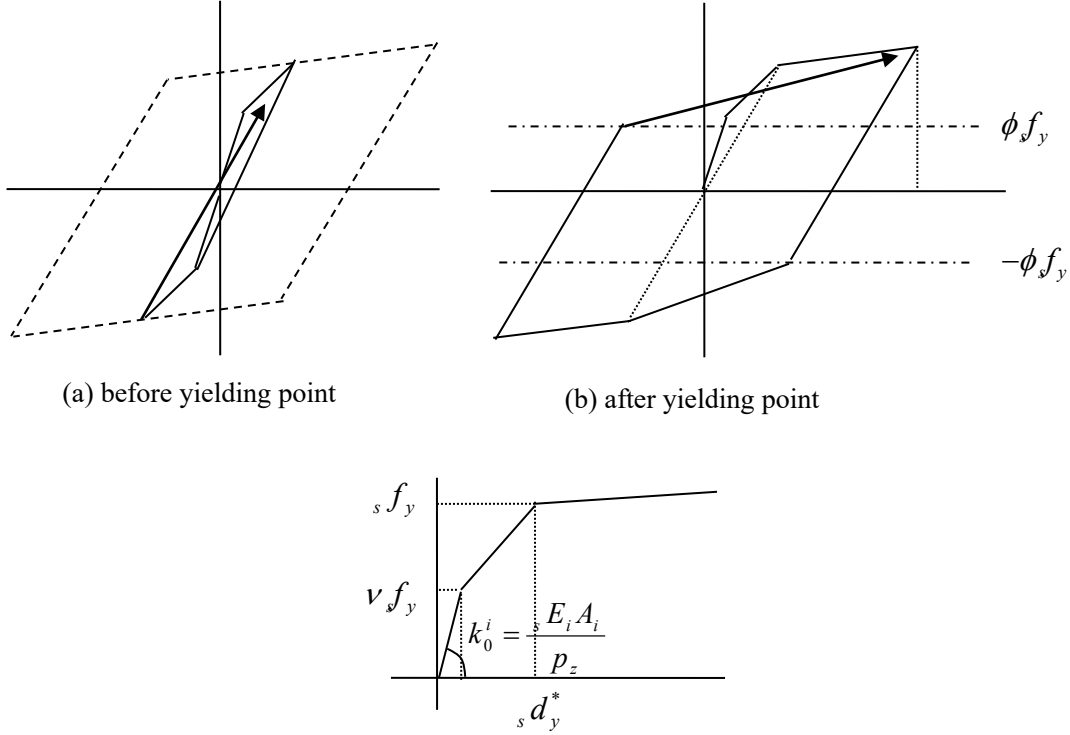


Figure A-1-1 Normal tri-linear model for steel spring

The hysteresis of steel spring has the degradation point at the forces,  $\nu f_y$  and  $\phi f_y$ , where  $\nu$  and  $\phi$  are the arbitrary parameters ( $\nu < 1$ ,  $\phi < 1$ ). The STERA\_3D Program adopts the values as:

$$\nu = 1/3, \quad \phi = 0.5 \quad (\text{A1-1})$$

Then, the yield deformation,  $s d_y^*$ , may be obtained by Equations (3-2-31) and (3-2-13) considering the reduction factor  $\gamma$ .

$$s d_y = \frac{\phi_y^* x_s}{1 + \frac{N_0 + 2 s f_y}{2 s f_y + 2 c f_y}} \quad (\text{A1-2})$$

$$\phi_y^* = \left( \frac{1}{\alpha_y} - \frac{1}{\gamma} \right) \frac{M_y}{k_0} \quad (\text{A1-3})$$

## b) Concrete spring

The hysteresis of concrete spring is also defined as tri-linear hysteresis model as shown in Figure 3-2-16. After compression yielding, strength degradation is considered by reducing the strength of the target point in reloading stage.

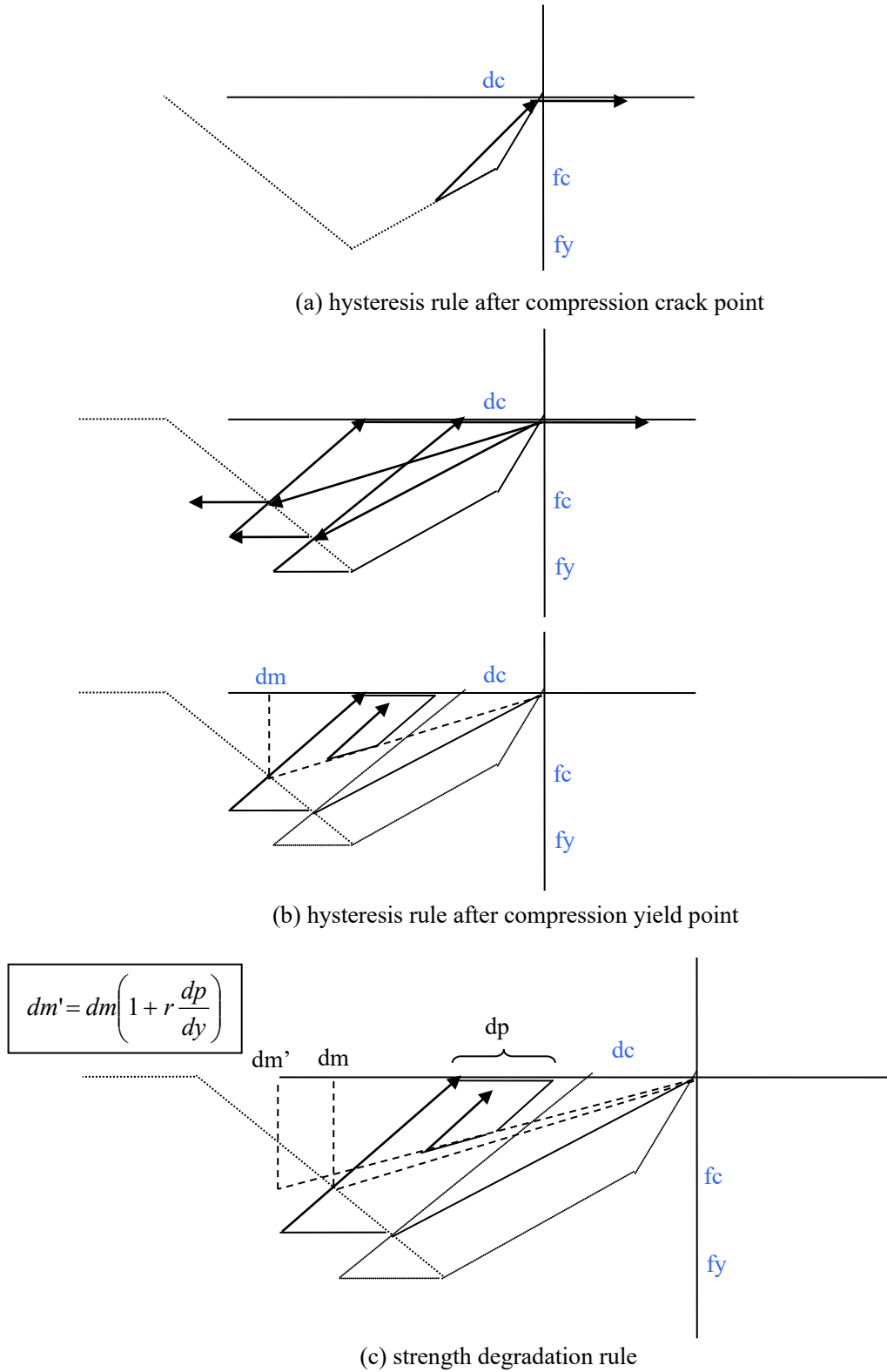


Figure A-1-2 Tri-linear hysteresis model for concrete spring

**Reference:** FRAME-D manual, Tohoku University, 1983 (in Japanese)

$$SE = \frac{|FRX| + |FRN|}{|DRX| + |DRN|}$$

$$S1 = 2 \times SE$$

$$S2 = 1.2 \times SE$$

$$FL1 = FL2 = 0.15 \times FRX$$

$$FL3 = FL4 = 0.15 \times FRN$$

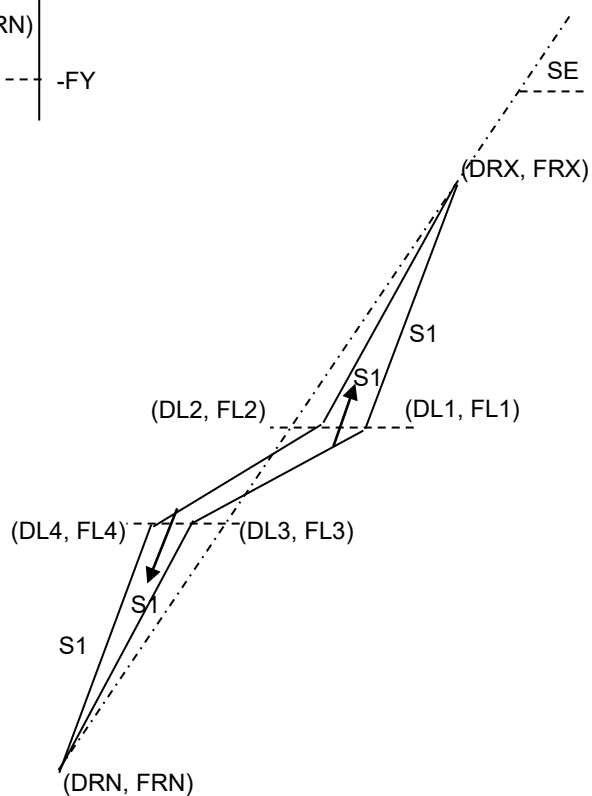


Figure A-2- Poly-linear slip hysteresis model for shear spring

### 3.3 Wall

#### 3.3.1 RC Wall

##### a) Section properties

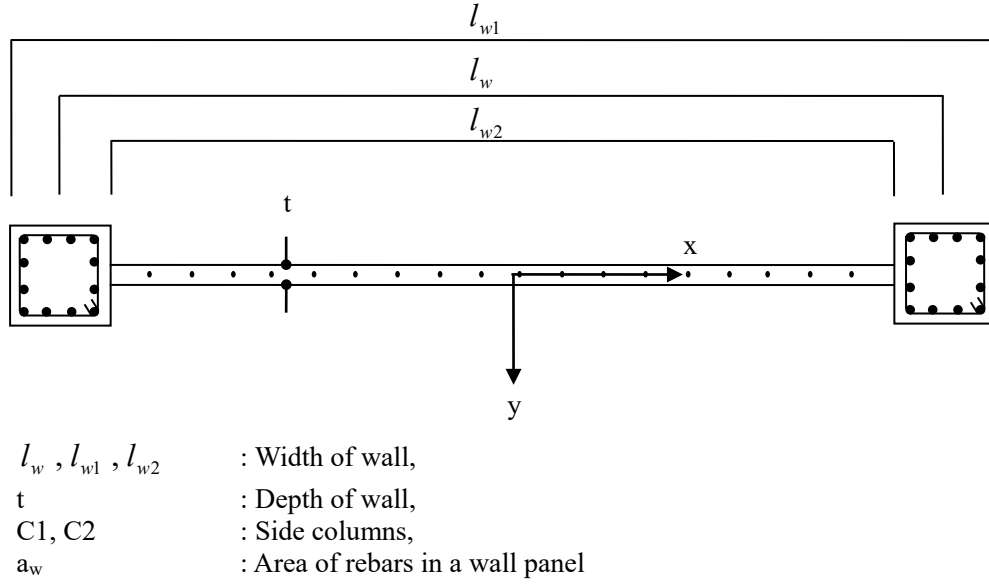


Figure 3-3-1 Wall Section

Area of section to calculate axial deformation

$$A_N = A_{N,C1} + A_{N,C2} + t l_{w2} + (n_E - 1)(a_w) \quad (3-3-1)$$

where,

$A_{N,C1}, A_{N,C2}$  : Area of section of side columns for axial deformation

$n_E = E_s / E_c$  : Ratio of Young's modulus between steel ( $E_s$ ) and concrete ( $E_c$ )

Area of section to calculate shear deformation

$$A_S = A_{S,C1} + A_{S,C2} + t l_{w2} / \kappa, \quad \kappa = 1.2 \quad (3-3-2)$$

where,

$A_{S,C1}, A_{S,C2}$  : Area of section of side columns for shear deformation

Moment of inertia around the center of the section

$$I_y = I_{y,C1} + I_{y,C2} + \frac{t \cdot l_{w2}^3}{12} + A_{N,C1} \left( \frac{l_{w1}}{2} \right)^2 + A_{N,C2} \left( \frac{l_{w1}}{2} \right)^2 \quad (3-3-3)$$

where,

$I_{y,C1}, I_{y,C2}$  : Moment of inertia of side columns

## b) Nonlinear bending spring

For the out of wall direction, each side columns behave independently in the same way as the column element. Therefore, we discuss here only the hysteresis model in the wall panel direction. Hysteresis model of nonlinear bending spring is defined as the moment-rotation relationship under the symmetry loading in Figure 3-3-5. The initial stiffness of the nonlinear spring is supposed to be infinite, however, in numerical calculation, a large enough value is used for the stiffness.

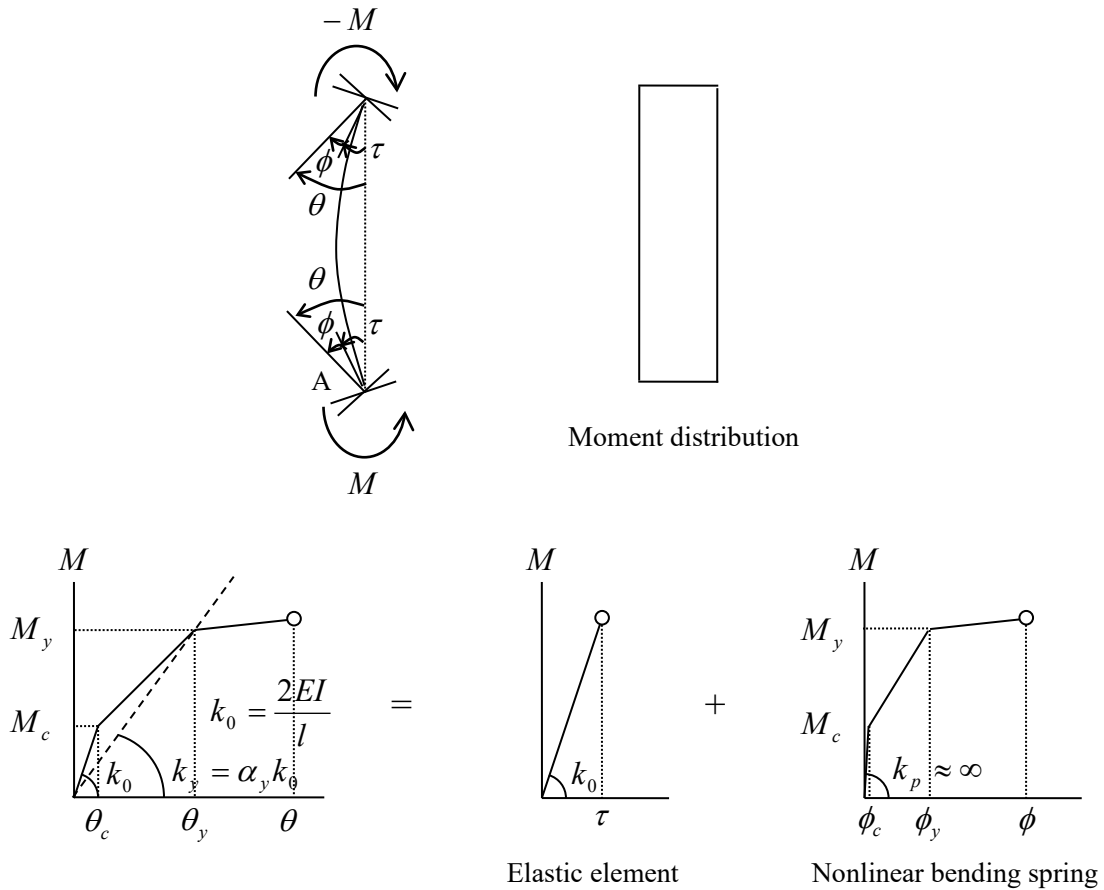


Figure 3-3-2 Moment – rotation relationship at bending spring

The yield moment,  $M_y$  is obtained from the equilibrium condition in Figure 3-3-6 as,

$$M_y = a_s \sigma_y l_w + 0.5 a_w \sigma_{wy} l_w + 0.5 N l_w \quad (3-3-4)$$

where,

- $a_s$  : Total area of rebar in the side column
- $\sigma_y$  : Strength of rebar in the side column
- $a_w$  : Total area of vertical rebar in the wall panel
- $\sigma_{wy}$  : Strength of rebar in the wall panel
- $N$  : Axial load from the dead load

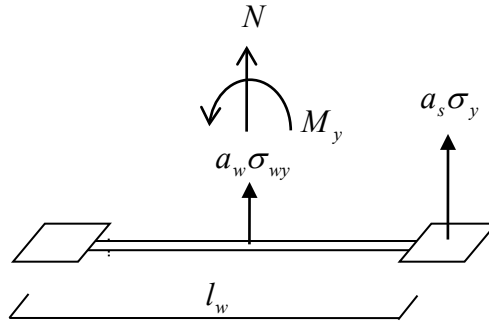


Figure 3-3-3 Equilibrium condition under yielding moment

The crack moment,  $M_c$  is assumed to be,

$$M_c = 0.3M_y \quad (3-3-5)$$

The tangential stiffness at the yield point,  $k_y$ , is obtained from the following equation:

$$k_y = 0.2K_0 \quad (3-3-6)$$

The yield rotation of the nonlinear bending beam,  $\phi_y$ , is then obtained from,

$$\phi_y = \left( \frac{1}{\alpha_y} - 1 \right) \frac{M_y}{K_0} \quad (3-3-7)$$

where, the stiffness degradation factor,  $\alpha_y$ , is assumed as,

$$\alpha_y = 0.02 \quad (3-3-8)$$

**Case 1: In the case that bending springs are independently defined**

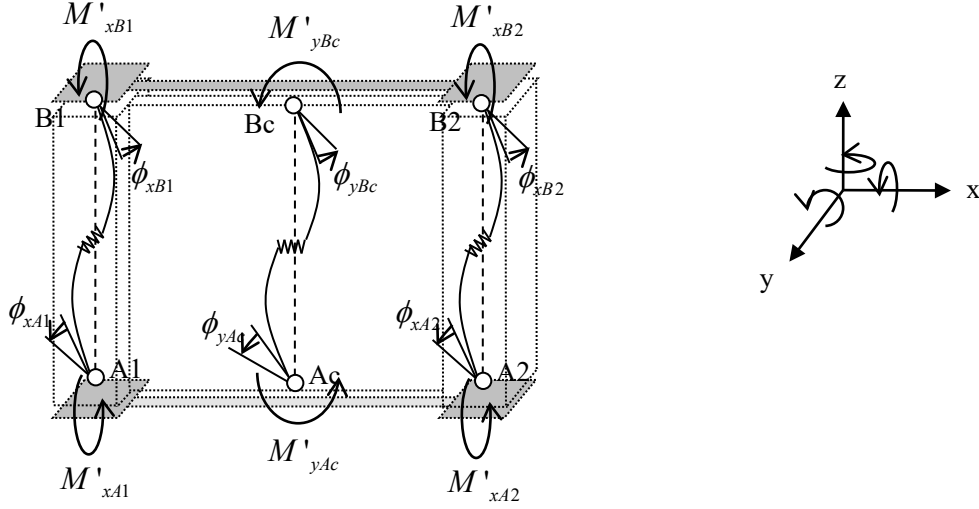


Figure 3-3-4 Nonlinear bending springs in the wall

The rotational displacement vector of the nonlinear bending spring is defined independently,

$$\begin{aligned}
 \phi_{xA1} &= f_{xA1} M'_{xA1}, \quad \phi_{xB1} = f_{xB1} M'_{xB1} && \text{in y-direction at Side Column 1} \\
 \phi_{xA2} &= f_{xA2} M'_{xA2}, \quad \phi_{xB2} = f_{xB2} M'_{xB2} && \text{in y-direction at Side Column 2} \\
 \phi_{yAc} &= f_{yAc} M'_{yAc}, \quad \phi_{yBc} = f_{yBc} M'_{yBc} && \text{in x-direction at center Wall panel}
 \end{aligned} \tag{3-3-9}$$

where,  $f_{xA1}$ ,  $f_{xB1}$ ,  $f_{xA2}$ ,  $f_{xB2}$ , and  $f_{yAc}$ ,  $f_{yBc}$  are the flexural stiffness of nonlinear bending springs at side columns and the center wall panel of the element, and

$$\begin{aligned}
 f_{xA1} &= 1/k_{xA1}, & f_{xB1} &= 1/k_{xB1} \\
 f_{xA2} &= 1/k_{xA2}, & f_{xB2} &= 1/k_{xB2} \\
 f_{yAc} &= 1/k_{yAc}, & f_{yBc} &= 1/k_{yBc}
 \end{aligned} \tag{3-3-10}$$

$$\begin{Bmatrix} \phi_{yAc} \\ \phi_{xA1} \\ \phi_{xA2} \\ \varepsilon_{zAc} \\ \phi_{yBc} \\ \phi_{xB1} \\ \phi_{xB2} \\ \varepsilon_{zBc} \end{Bmatrix} = \begin{bmatrix} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{bmatrix} \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix} = \begin{bmatrix} f_{yAc} & & & & & & \\ & f_{xA1} & & & & & \\ & & f_{xA2} & & & & \\ & & & 0 & & & \\ & & & & f_{yBc} & & \\ & & & & & f_{xB1} & \\ & & & & & & f_{xB2} \\ & & & & & & & 0 \end{bmatrix} \begin{Bmatrix} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{Bmatrix} \tag{3-3-11}$$

The hysteresis model for  $M - \phi$  relationship is the degrading tri-linear slip model as used for the hysteresis model of the bending springs of the RC beam.

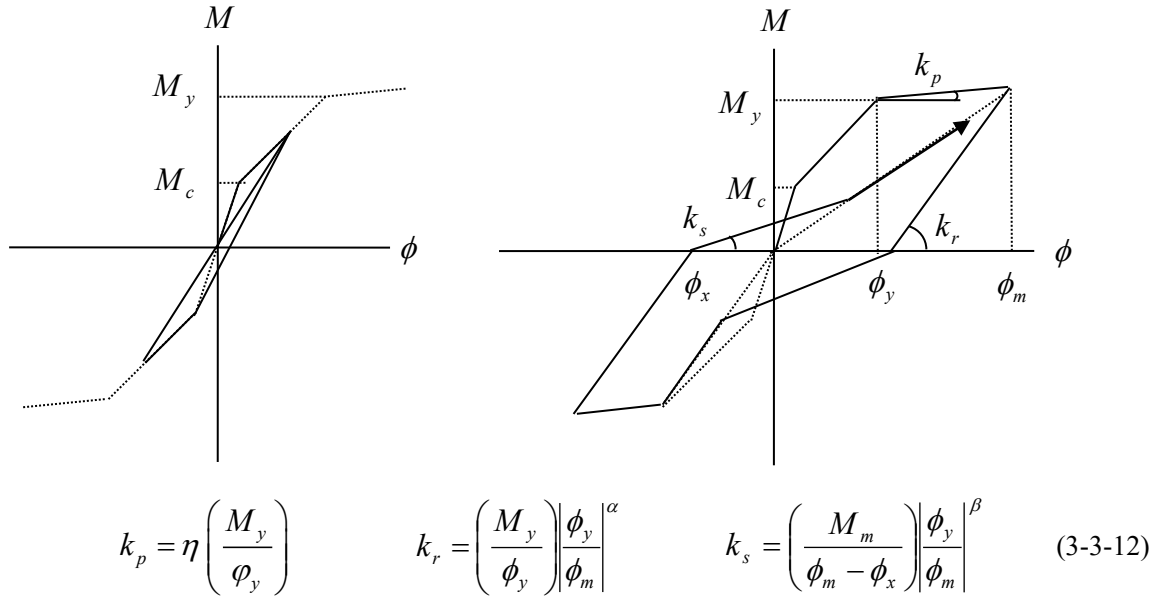


Figure 3-3-5 Degrading Tri-linear Slip Model  
( $\alpha=0.5$ ,  $\beta=0.0$  and  $\eta=0.001$  as default values)

**Case 2: In the case that nonlinear interaction between moment and axial components is considered**

To consider nonlinear interaction among  $M_x - M_y - N_z$ , the nonlinear bending spring at the member end is constructed from the nonlinear vertical springs arranged in the member section as shown in Figure 3-3-2.

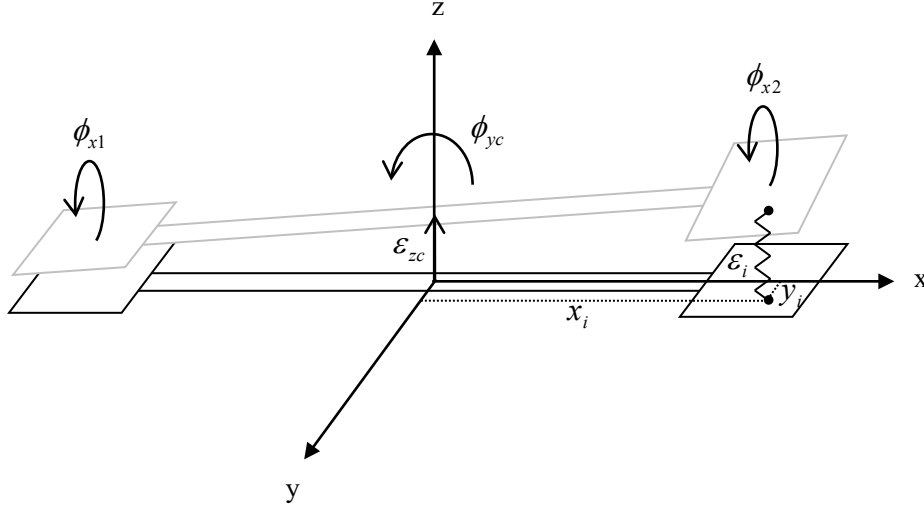


Figure 3-3-6 Nonlinear bending springs

Displacement of the i-th nonlinear axial spring is,

$$\begin{aligned}
 \varepsilon_i &= \varepsilon_{zc} + x_i \phi_{yc} && \text{in a wall panel} \\
 \varepsilon_i &= \varepsilon_{zc} - y_i \phi_{x1} + x_i \phi_{yc} && \text{in a side column 1} \\
 \varepsilon_i &= \varepsilon_{zc} - y_i \phi_{x2} + x_i \phi_{yc} && \text{in a side column 2}
 \end{aligned}
 \tag{3-3-13}$$

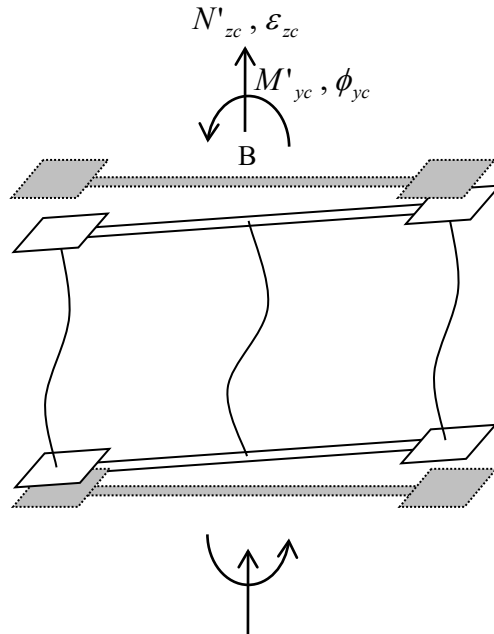


Figure 3-3-7 Equilibrium condition in the wall panel direction

In the wall panel direction, all vertical springs in the nonlinear section are assumed to work against the moment and the axial force. The equilibrium conditions are,

$$\begin{aligned}
 M'_{yc} &= \sum_i^{Nc} k_i \varepsilon_i x_i + \sum_i^{N1} k_i \varepsilon_i x_i + \sum_i^{N2} k_i \varepsilon_i x_i \\
 &= \sum_i^{Nc} k_i (\varepsilon_{zc} + x_i \phi_{yc}) x_i + \sum_i^{N1} k_i (\varepsilon_{zc} - y_i \phi_{x1} + x_i \phi_{yc}) x_i + \sum_i^{N2} k_i (\varepsilon_{zc} - y_i \phi_{x2} + x_i \phi_{yc}) x_i \\
 &= \left[ \begin{array}{ccc} \sum_i^{Nc+N1+N2} k_i x_i^2 & - \sum_i^{N1} k_i x_i y_i & - \sum_i^{N2} k_i x_i y_i \\ \sum_i^{Nc+N1+N2} k_i x_i & & \end{array} \right] \left\{ \begin{array}{c} \phi_{yc} \\ \phi_{x1} \\ \phi_{x2} \\ \varepsilon_{zc} \end{array} \right\}
 \end{aligned}
 \tag{3-3-14}$$

$$\begin{aligned}
 N'_{zc} &= \sum_i^{Nc} k_i \varepsilon_i + \sum_i^{N1} k_i \varepsilon_i + \sum_i^{N2} k_i \varepsilon_i \\
 &= \sum_i^{Nc} k_i (\varepsilon_{zc} + x_i \phi_{yc}) + \sum_i^{N1} k_i (\varepsilon_{zc} - y_i \phi_{x1} + x_i \phi_{yc}) + \sum_i^{N2} k_i (\varepsilon_{zc} - y_i \phi_{x2} + x_i \phi_{yc}) \\
 &= \left[ \begin{array}{ccc} \sum_i^{Nc+N1+N2} k_i x_i & - \sum_i^{N1} k_i y_i & - \sum_i^{N2} k_i y_i \\ \sum_i^{Nc+N1+N2} k_i & & \end{array} \right] \left\{ \begin{array}{c} \phi_{yc} \\ \phi_{x1} \\ \phi_{x2} \\ \varepsilon_{zc} \end{array} \right\}
 \end{aligned}
 \tag{3-3-15}$$

where,  $Nc$ ,  $N1$  and  $N2$  are the number of vertical springs in a wall panel, side column 1 and side column 2, respectively.

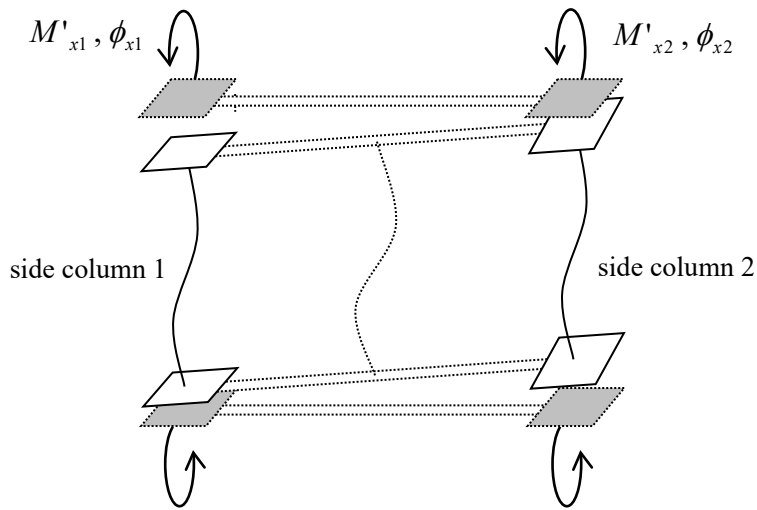


Figure 3-3-8 Equilibrium condition in the out of wall direction

In the out of wall direction, we establish the equilibrium condition for each side column independently. The equilibrium condition for the side column 1 is,

$$\begin{aligned}
M'_{x1} &= -\sum_i^{N1} k_i \varepsilon_i y_i \\
&= -\sum_i^{N1} k_i (\varepsilon_{zc} - y_i \phi_{x1} + x_i \phi_{yc}) y_i \\
&= \left[ -\sum_i^{N1} k_i x_i y_i \quad \sum_i^{N1} k_i y_i^2 \quad 0 \quad -\sum_i^{N1} k_i y_i \right] \begin{Bmatrix} \phi_{yc} \\ \phi_{x1} \\ \phi_{x2} \\ \varepsilon_{zc} \end{Bmatrix}
\end{aligned} \tag{3-3-16}$$

Also, for the side column 2,

$$\begin{aligned}
M'_{x2} &= -\sum_i^{N2} k_i \varepsilon_i y_i \\
&= -\sum_i^{N2} k_i (\varepsilon_{zc} - y_i \phi_{x1} + x_i \phi_{yc}) y_i \\
&= \left[ -\sum_i^{N2} k_i x_i y_i \quad 0 \quad \sum_i^{N2} k_i y_i^2 \quad -\sum_i^{N2} k_i y_i \right] \begin{Bmatrix} \phi_{yc} \\ \phi_{x1} \\ \phi_{x2} \\ \varepsilon_{zc} \end{Bmatrix}
\end{aligned} \tag{3-3-17}$$

In a matrix form

$$\begin{Bmatrix} M'_{yc} \\ M'_{x1} \\ M'_{x2} \\ N'_{zc} \end{Bmatrix} = \begin{bmatrix} \sum_i^{Nc+N1+N2} k_i x_i^2 & -\sum_i^{N1} k_i x_i y_i & -\sum_i^{N2} k_i x_i y_i & \sum_i^{Nc+N1+N2} k_i x_i \\ -\sum_i^{N1} k_i x_i y_i & \sum_i^{N1} k_i y_i^2 & 0 & -\sum_i^{N1} k_i y_i \\ -\sum_i^{N2} k_i x_i y_i & 0 & \sum_i^{N2} k_i y_i^2 & -\sum_i^{N2} k_i y_i \\ \sum_i^{Nc+N1+N2} k_i x_i & -\sum_i^{N1} k_i y_i & -\sum_i^{N2} k_i y_i & \sum_i^{Nc+N1+N2} k_i \end{bmatrix} \begin{Bmatrix} \phi_{yc} \\ \phi_{x1} \\ \phi_{x2} \\ \varepsilon_{zc} \end{Bmatrix} = [k_p] \begin{Bmatrix} \phi_{yc} \\ \phi_{x1} \\ \phi_{x2} \\ \varepsilon_{zc} \end{Bmatrix} \tag{3-3-18}$$

Therefore

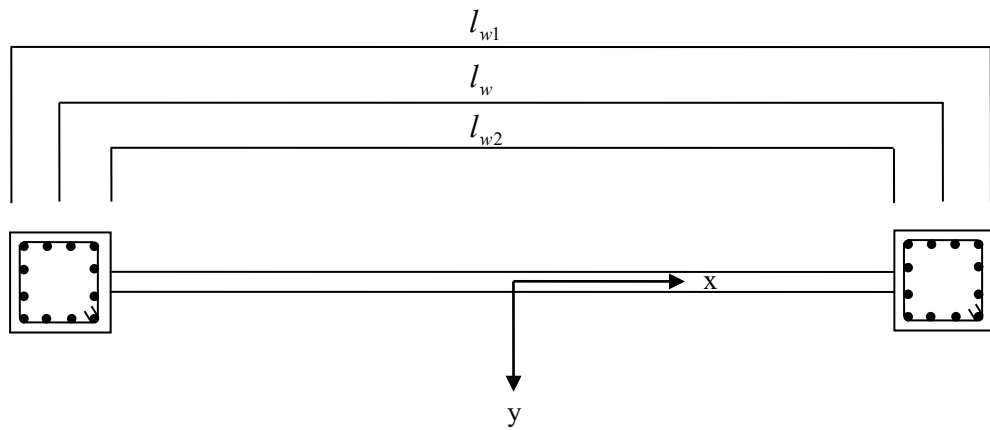
$$\begin{Bmatrix} \phi_{yc} \\ \phi_{x1} \\ \phi_{x2} \\ \varepsilon_{zc} \end{Bmatrix} = [k_p]^{-1} \begin{Bmatrix} M'_{yc} \\ M'_{x1} \\ M'_{x2} \\ N'_{zc} \end{Bmatrix} = [f_p] \begin{Bmatrix} M'_{yc} \\ M'_{x1} \\ M'_{x2} \\ N'_{zc} \end{Bmatrix} \tag{3-3-19}$$

For both ends

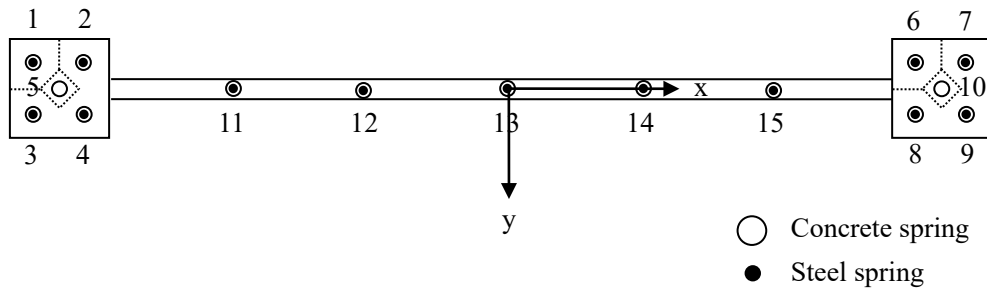
$$\left\{ \begin{array}{c} \phi_{yAc} \\ \phi_{xA1} \\ \phi_{xA2} \\ \varepsilon_{zAc} \\ \phi_{yBc} \\ \phi_{xB1} \\ \phi_{xB2} \\ \varepsilon_{zBc} \end{array} \right\} = \left[ \begin{array}{cc} [f_{pA}] & 0 \\ 0 & [f_{pB}] \end{array} \right] \left\{ \begin{array}{c} M'_{yAc} \\ M'_{xA1} \\ M'_{xA2} \\ N'_{zAc} \\ M'_{yBc} \\ M'_{xB1} \\ M'_{xB2} \\ N'_{zBc} \end{array} \right\} \quad (3-3-20)$$

### c) Nonlinear vertical springs

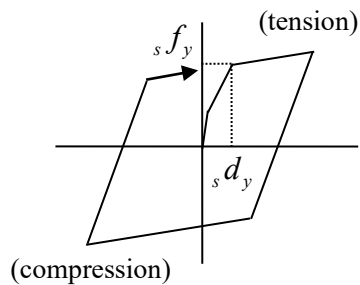
The nonlinear bending spring is constructed from the nonlinear vertical springs arranged in the member section as shown in Figure 3-3-6. This model is based on the concept of “Multi-spring model” and modified for the wall element by Saito et.al. The vertical springs in the side columns are determined independently in the same way as the Multi-spring models of columns. The wall panel section is divided in 5 areas, and a steel springs and a concrete spring are arranged at the center of each area.



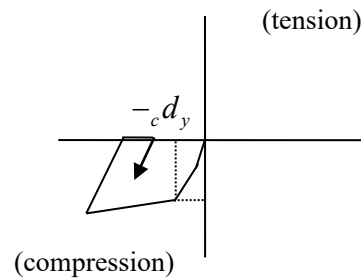
(a) Original column section



(b) Multi-spring model



(c) Hysteresis of steel spring



(d) Hysteresis of concrete spring

Figure 3-3-9 Nonlinear vertical springs

### Strength of steel spring in wall panel

The strength of the steel spring in the wall panel is one-fifth of total strength of rebars in the section,

$${}_s f_y = \frac{a_w \sigma_{wy}}{5} \quad (3-3-21)$$

where,

- $a_w$  : Total area of vertical rebar in the wall panel  
 $\sigma_{wy}$  : Strength of rebar in the wall panel

### Strength of concrete spring in wall panel

The strength of the concrete spring in the wall panel is one-fifth of total strength of concrete in the section,

$${}_c f_y = \frac{0.85 A_p \sigma_B}{5} \quad (3-3-22)$$

where,

- $A_p$  : Total area of wall panel section  
 $\sigma_B$  : Compression strength of concrete

### Yield displacement of vertical spring in wall panel

The yield displacements of steel and concrete springs in the wall panel are assumed to be the same as those of the springs in the side columns.

### d) Nonlinear shear spring

There are three nonlinear shear springs in x direction in wall panel and y direction in side columns. Hysteresis model of the nonlinear shear springs is the same as that in the beam element in Figure 3-1-4.

### Yield shear force

The yield shear force,  $Q_y$  is calculated as,

$$Q_y = \left\{ \frac{0.053 p_t^{0.23} (\sigma_B + 18)}{M/(QD) + 0.12} + 0.85 \sqrt{p_w \cdot \sigma_{wy}} + 0.1 \sigma_0 \right\} b \cdot j \quad (3-3-23)$$

where,

- $b$  : Equivalent thickness of the wall ( $\sum A/l_{w1}$ )  
 $j$  : Distance between the centroids of tension and compression forces ( $0.8l_{w1}$ )  
 $p_t$  : Tensile reinforcement ratio ( $100a_t/(b \cdot l)$ )(%)  
 $\sigma_B$  : Compression strength of concrete ( $N/\sum A$ )  
 $M/(QD)$  : Shear span-to-depth ratio ( $=h/(2l_{w1})$ )  
 $= 1 (h/(2l_{w1}) < 1), \quad 3 (h/(2l_{w1}) > 3)$   
 $p_w$  : Shear reinforcement ratio

$\sigma_{wy}$  : Strength of shear reinforcement  
 $\sigma_0$  : Axial stress of the column

#### Crack shear force

The crack shear force is,  $Q_c$ , is assumed as,

$$Q_c = \frac{Q_y}{3} \quad (3-3-24)$$

#### Ultimate shear force

The crack shear force is,  $Q_u$ , is assumed as,

$$Q_u = Q_c \quad (3-3-25)$$

#### Crack shear deformation

The crack shear deformation is obtained as,

$$s_c = \gamma_c l, \quad \gamma_c = \frac{Q_c}{GA} \quad (3-3-26)$$

#### Yield shear displacement

The yield shear deformation is assumed as,

$$s_y = \gamma_y l, \quad \gamma_y = \frac{1}{250} \quad (3-3-27)$$

#### Ultimate shear displacement

The ultimate shear deformation is assumed as,

$$s_u = \gamma_u l, \quad \gamma_u = \frac{1}{100} \quad (3-3-28)$$

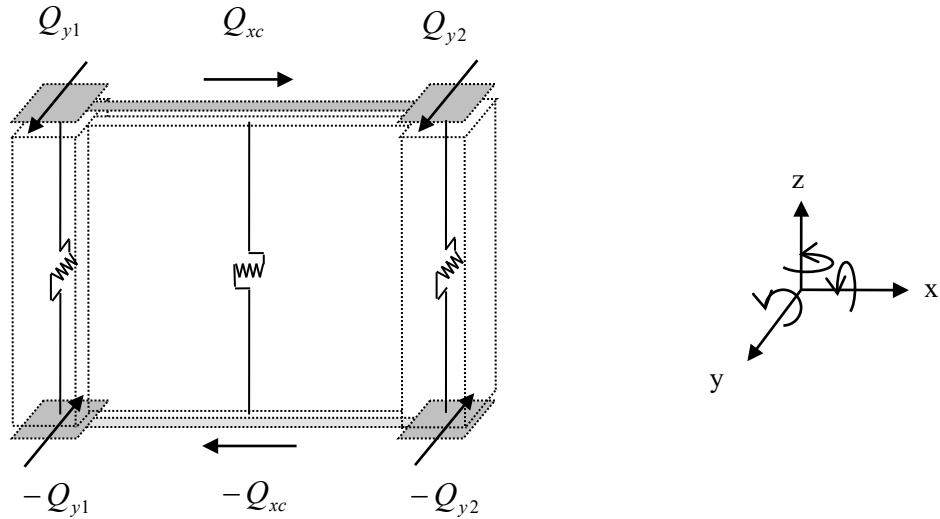


Figure 3-3-10 Nonlinear shear springs in the wall

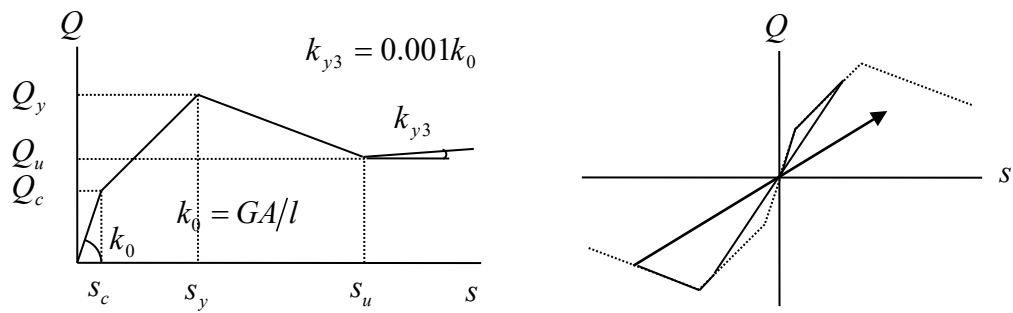
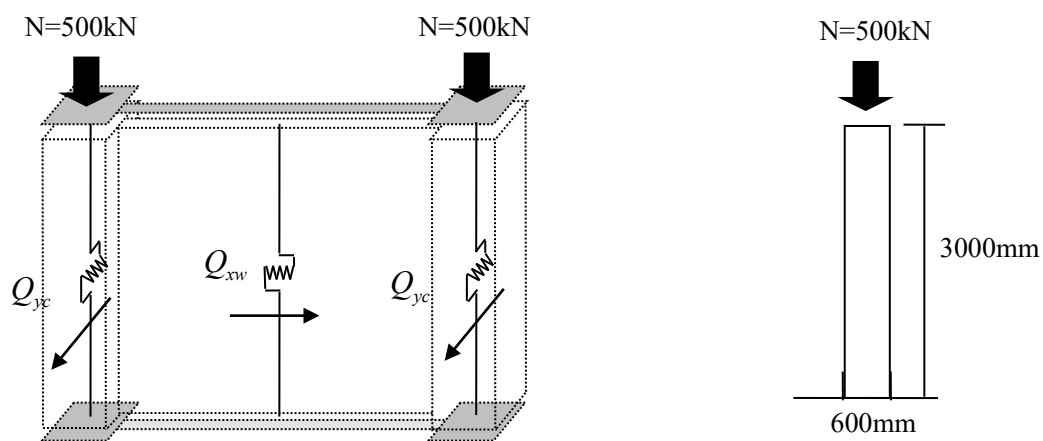
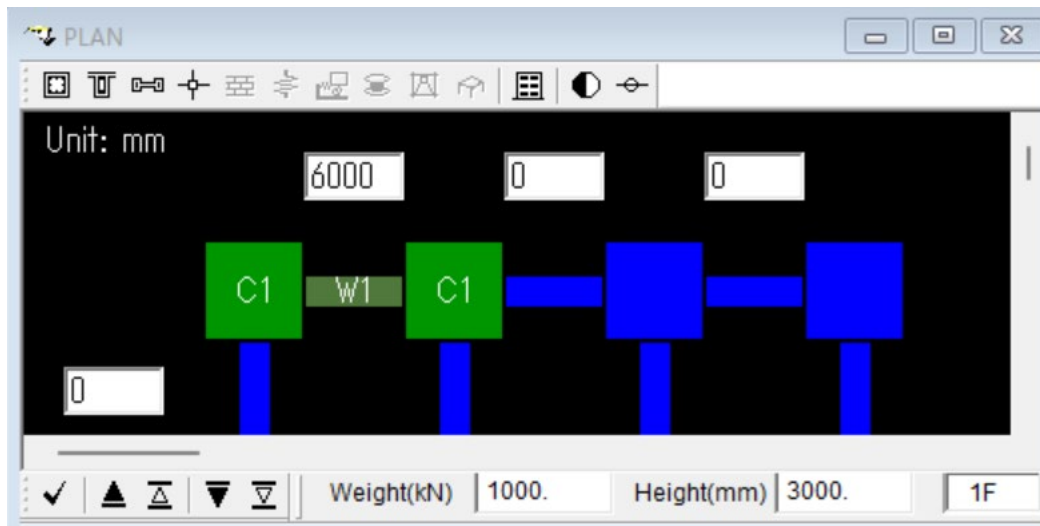


Figure 3-3-11 Force–deformation relationship of shear spring

Example)



Shear strength of the side column in y-direction  $Q_{yc} = 479.2$  (kN) (see Column element)

Wall Editor

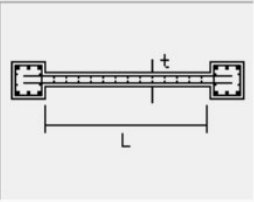
WALL

Type

W1  
W2  
W3  
W4  
W5  
W6  
W7  
W8  
W9  
W10  
W11  
W12  
W13

Size

t (mm)  
150



Shear Reinforcement in a Panel

SD (N/mm2)

2 - D13 - @ 150 295

Copy

Concrete (N/mm2)

Fc 24

$$Q_y = \left\{ \frac{0.053 p_t^{0.23} (\sigma_B + 18)}{M / (QD) + 0.12} + 0.85 \sqrt{p_w \cdot \sigma_{wy}} + 0.1 \sigma_0 \right\} b \cdot j$$

where,

$$b = \sum A / l_{w1} = 1652528 / 660 = 250 \text{ (mm)}, \quad j = 0.8 l_{w1} = 5280 \text{ (mm)}$$

$$p_t = 100 a_t / (b \cdot l) = 100 (3096.8) / 1652528 = 0.187 \text{ (%)}, \quad \sigma_B = 240 \text{ (N/mm}^2\text{)},$$

$$M / (QD) \approx h / (2 l_{w1}) = 3000 / 13200 < 1 \rightarrow = 1.0$$

$$p_w = 100 \cdot a_w / (b \cdot x) = 0.0067, \quad a_w = 2 \text{ D13} = 253 \text{ (mm}^2\text{)}, \quad x = 150 \text{ (mm)}$$

$$\sigma_{wy} = 1.1 (295) = 324.5 \text{ (N/mm}^2\text{)}, \quad \sigma_0 = N / \sum A = 1000000 / 1652528 = 0.605 \text{ (N/mm}^2\text{)}$$

$$Q_{yw} = 3458.37 \text{ (kN)}$$

### e) Modification of initial stiffness of nonlinear springs

The same modification can be done for the nonlinear springs of wall element as described for those of beam and column elements by reducing the initial stiffness of the nonlinear spring and increasing the stiffness of the elastic element as shown in the following figure:

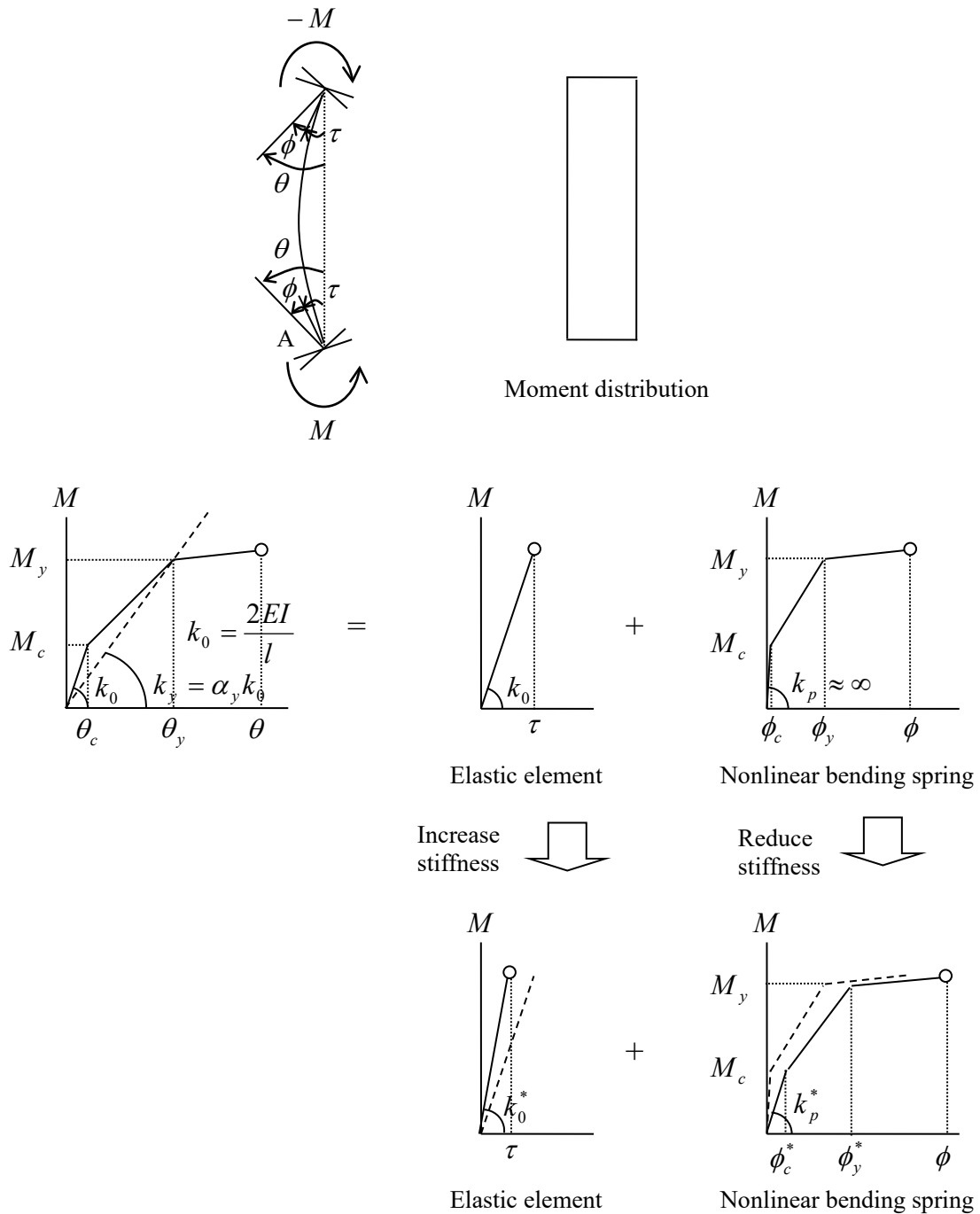


Figure 3-3-12 Modification of moment – rotation relationship

Introducing the concept of “plastic zones”, the initial stiffness of the  $i$ -th multi-spring can be expressed as,

$$k_0^i = \frac{E_i A_i}{p_z} \quad (3-3-29)$$

where  $E_i$  : the material young's modulus,  $A_i$  : the spring governed area, and  $p_z$  : the length of assumed plastic zone. When  $p_z \rightarrow 0$ , it represents the infinite stiffness for rigid condition.

In the same manner of beam and column elements, introducing the flexibility reduction factors,  $\gamma_0 (< 0)$ ,  $\gamma_1 (< 0)$ ,  $\gamma_2 (< 0)$ , the flexibility matrix of the elastic element is,

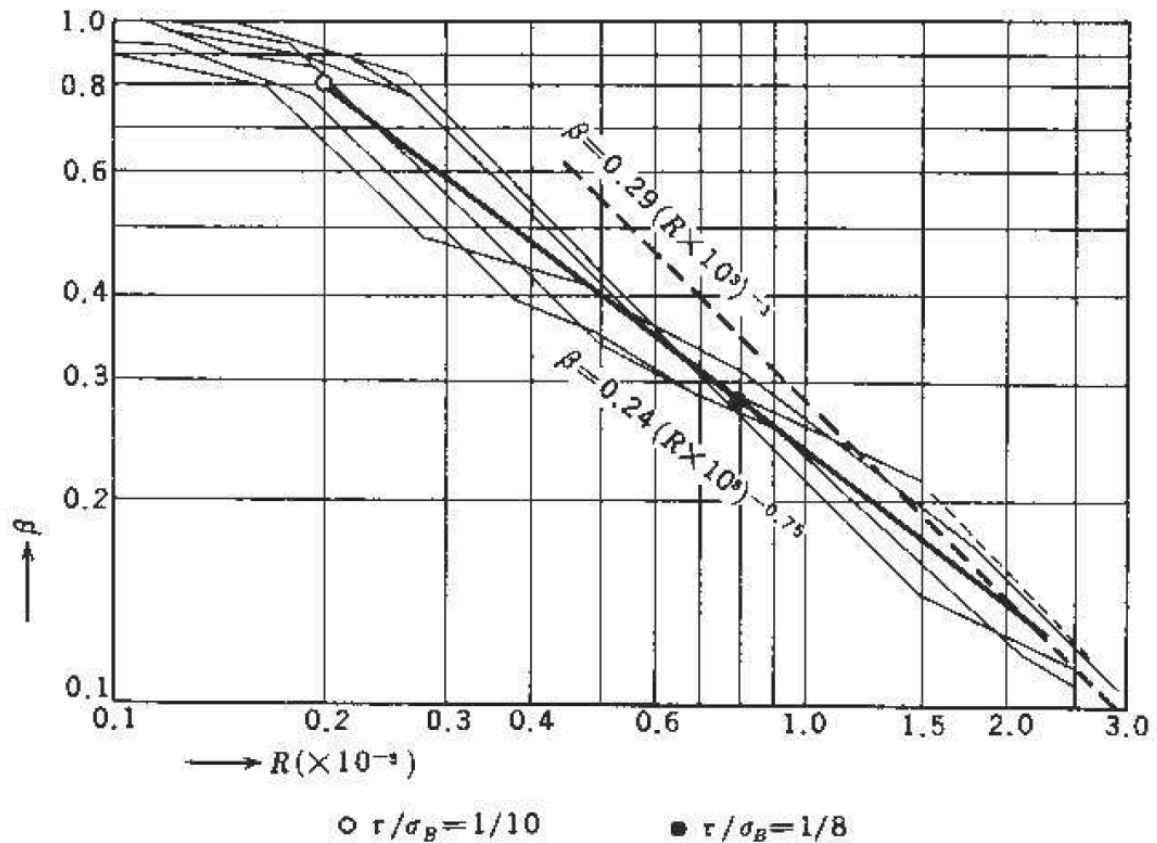
$$[f_w] = \begin{bmatrix} \gamma_1 \frac{l'}{3EI_c} & -\frac{l'}{6EI_c} & & & & \\ & \gamma_2 \frac{l'}{3EI_c} & & & & \\ & & \gamma_1 \frac{l'}{3EI_1} & -\frac{l'}{6EI_1} & & \\ & & & \gamma_2 \frac{l'}{3EI_1} & & \\ & & & & \gamma_1 \frac{l'}{3EI_2} & -\frac{l'}{6EI_2} \\ & & & & & \gamma_2 \frac{l'}{3EI_2} \\ & sym. & & & & \\ & & & & & & \gamma_0 \frac{l'}{EA_c} \end{bmatrix} \quad (3-3-30)$$

Also, adopting  $p_z = \frac{l'}{10}$  as discussed for beam and column elements, the reduction factors will be:

$$\gamma_1 = \gamma_2 = 0.7, \quad \gamma_0 = 0.8 \quad (3-3-31)$$

#### f) Reduction factor of shear stiffness

If shear cracking occurs in the reinforced concrete wall, the shear stiffness decreases. The following graph shows the test results of the relationship between the stiffness reduction factor  $\beta$  and the lateral drift angle  $R \times 10^{-3}$  (referred from “Standard for Structural Calculation of Reinforced Concrete Structure”, Architectural Institute of Japan).



For example, if the lateral drift angle is over than 1/1000, the reduction factor becomes less than 0.2. Therefore, STERA\_3D assumes the “Reduction Factor for Stiffness” is 0.2 in the default setting for the option of the RC wall element.

Wall Option Editor
×

WALL OPTION

1. Amplification Factor for Steel Strength [0, 2]

2. Reduction Factor for Stiffness [0, 1]

3. Reduction Factor for Strength [0, 1]

### 3.3.1 Direct Wall

Direct Wall identifies the force-displacement points in the back-bone curves of the nonlinear shear spring and the nonlinear bending spring.

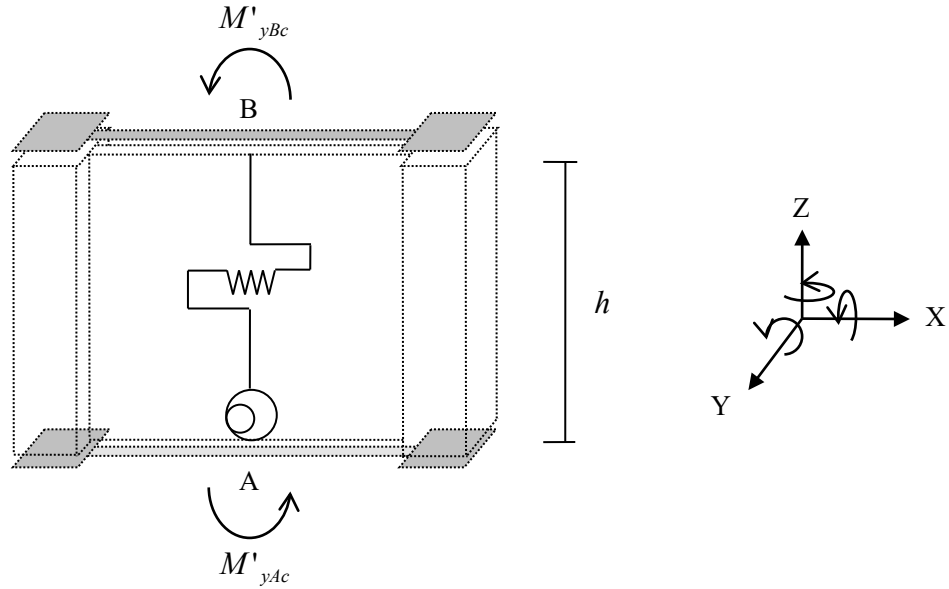


Figure 3-3-13 Element model for wall

Different types of hysteresis model are prepared for the force-deformation relationship of the spring.

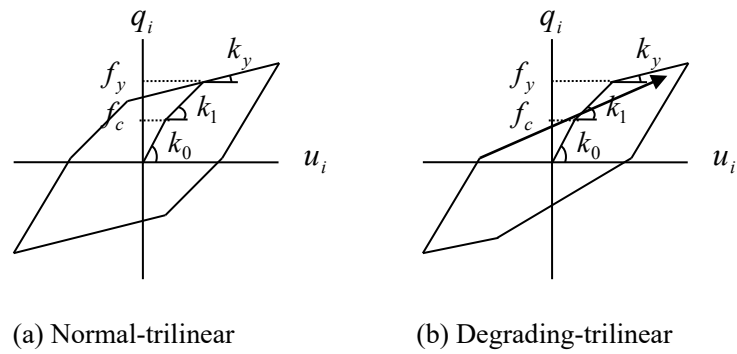


Figure 3-3-14 Hysteresis model of the shear and bending springs

### 3.3.2 Steel Wall (Brace)

#### a) Buckling of brace

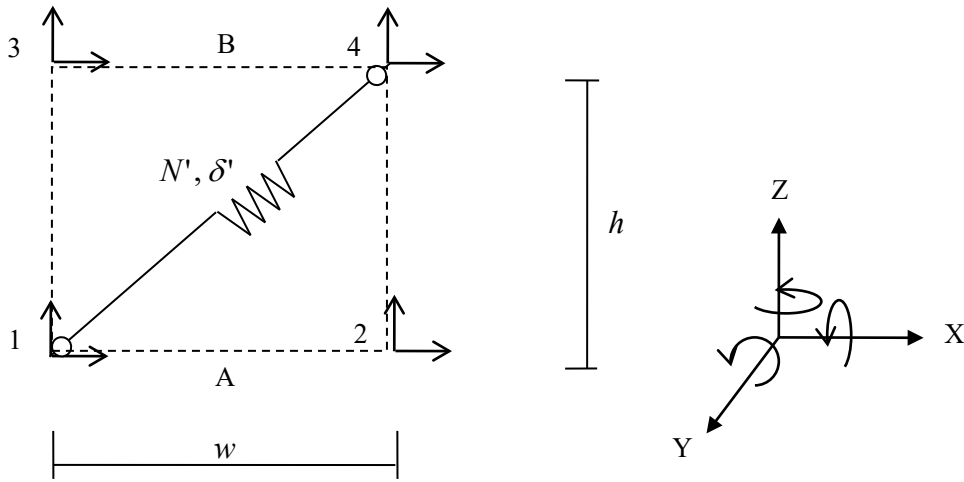


Figure 3-3-15 Element model for brace

Under the compression load, the stress of buckling failure is calculated theoretically as

$$\sigma_E = \frac{\pi^2 E}{\lambda^2},$$

where  $\lambda = \frac{L}{i}$ : slenderness ratio



If  $\sigma_E > \sigma_y$  (strength of steel), the compression failure will occur before buckling.

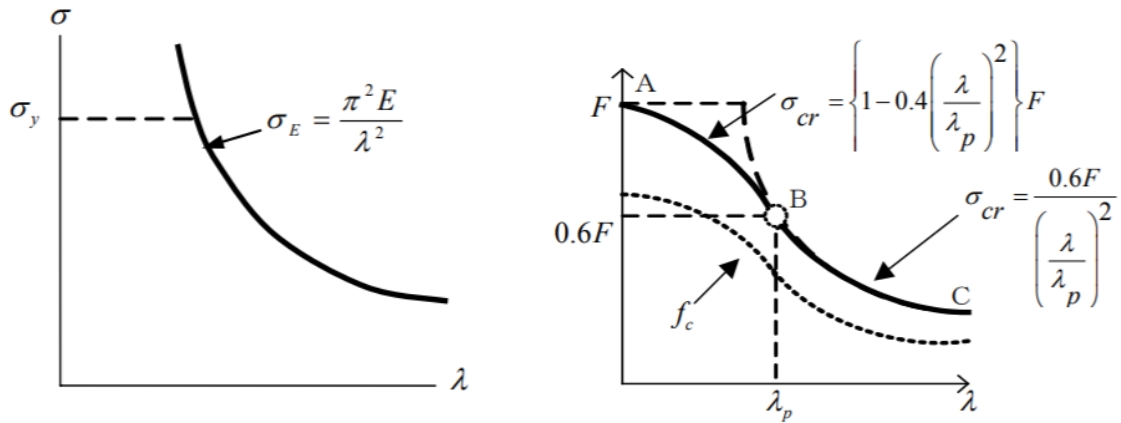


Figure 3-3-16 Relationship between buckling stress and slenderness ratio

The AIJ (Architectural Institute of Japan) guideline adopts the following equation for the stress of buckling.

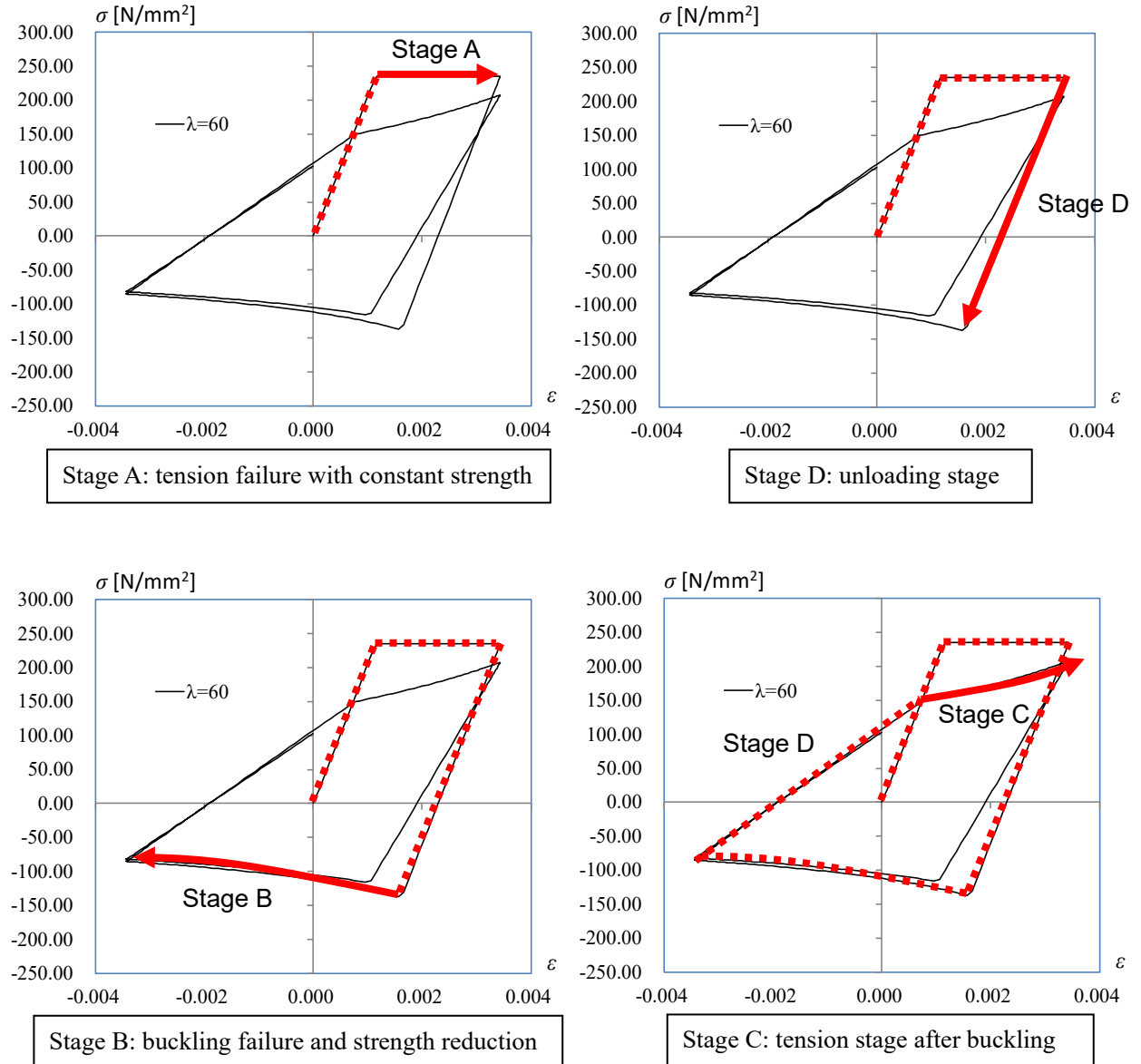
$$\sigma_{cr} = \left\{ 1 - 0.4 \left( \lambda / \lambda_p \right)^2 \right\} \sigma_y, \quad \text{for } \lambda \leq \lambda_p \quad (3-3-32)$$

$$\sigma_{cr} = \frac{0.6}{\left( \lambda / \lambda_p \right)} \sigma_y, \quad \text{for } \lambda > \lambda_p \quad (3-3-33)$$

where  $\lambda_p = \sqrt{\frac{\pi^2 E}{0.6 \sigma_y}}$  : Critical slenderness ratio

### b) Hysteresis model

The hysteresis model proposed by Wakabayashi et. al. is adopted in STERA\_3D (hereinafter referred to as Wakabayashi model). The model consists of four Stages A, B, C and D.



The compression curve (Stage B) and the tension curve (Stage C) are defined using the nondimensional strength and deformation as,

$$n = N / N_0 : \text{nondimensional strength}$$

$$\delta = \Delta / \Delta_0 : \text{nondimensional deformation}$$

where  $N$  : axial load,  $N_0 = A\sigma_y$  : axial strength (A: area,  $\sigma_y$  : yielding stress of steel)

$$\Delta$$
 : displacement,  $\Delta_0 = L\varepsilon_y = L\sigma_y / E$  : yield deformation

Both curves are assumed to be the following form

$$n = 1 / (a\delta + b)^r$$

where  $a, b$  : parameters of the function of nondimensional Euler load  $n_E = \sigma_E / \sigma_y = \pi^2 E / (\lambda^2 \sigma_y)$

### b-1) Compression Curve

Compression curve (Stage B) is defined from the following empirical formula,

$$n = 1 / (p_1 \delta + p_2)^{1/2}$$

where  $p_1 = \frac{10/n_E - 1}{3}$ ,  $p_2 = 4/n_E + 0.6$

Compression strength  $n_c$  is also on this curve, therefore,

$$n_c = 1 / (p_1 \delta_c + p_2)^{1/2}$$

or  $p_1 n_c^2 \delta_c + p_2 n_c - 1 = 0$

Since  $n_c = \frac{N_c}{N_0} = \frac{E(\Delta_c / L) A}{E(\Delta / L) A} = \frac{\Delta_c}{\Delta} = \delta$

Finally  $n_c$  is obtained by solving

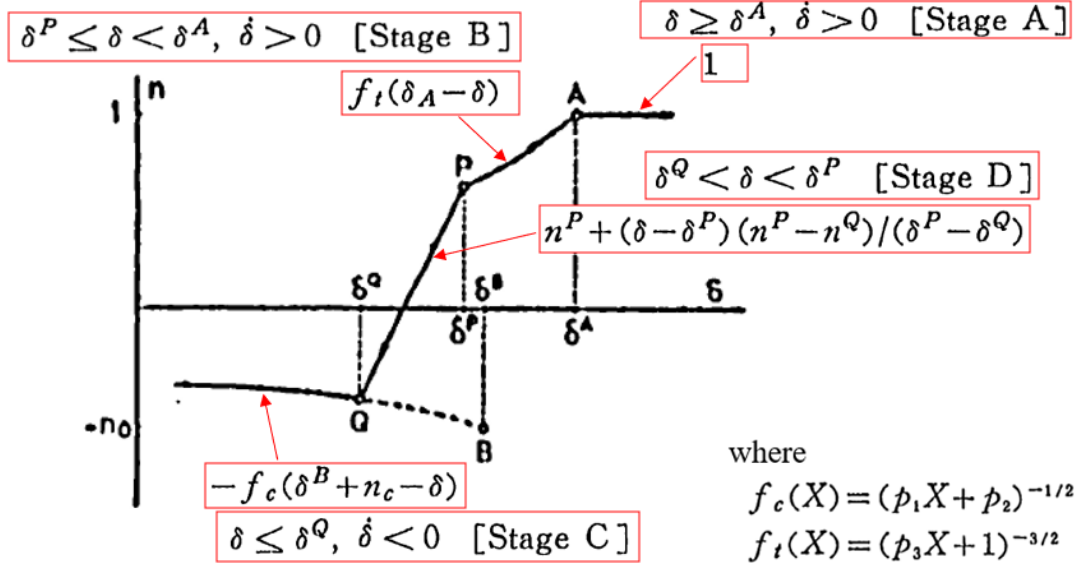
$$p_1 n_c^3 + p_2 n_c - 1 = 0$$

### b-2) Tension Curve

Tension curve (Stage C) is defined from the following empirical formula,

$$n = 1 / (p_3 \delta + 1)^{3/2}$$

where  $p_3 = \frac{1}{3.1 n_E + 1.4}$

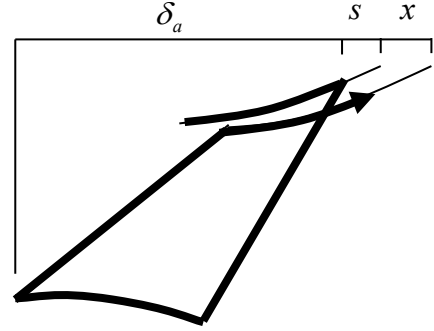


### b-3) Movement of Tension Curve

Movement of tension curve  $x$  is defined as follows:

$$x = \ln(q_1 \delta_a + 1) - q_2 s$$

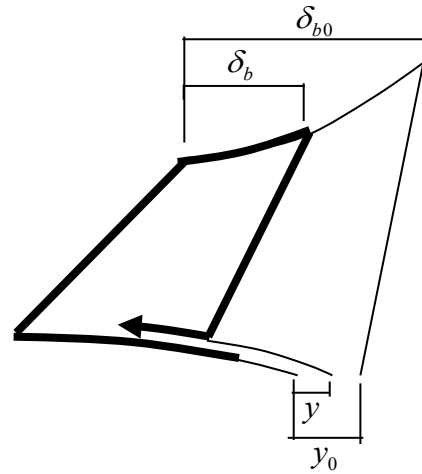
where  $q_1 = \frac{3-1/n_E}{10}, q_2 = 0.115/n_E + 0.36$



### b-4) Movement of Compression Curve

Movement of compression curve  $y$  is defined to satisfy the following relationship

$$\frac{y}{y_0} = \frac{\delta_b}{\delta_{b0}}$$

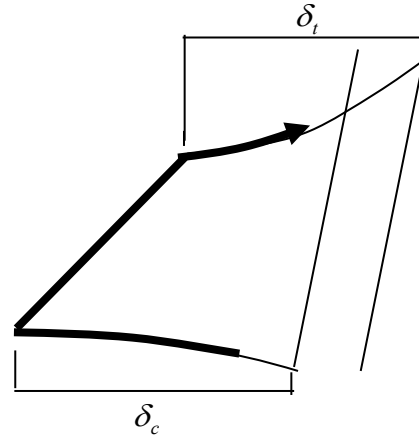


### b-5) Movement of Compression Curve

The point shifting from the unloading Stage D to Stage C is obtained by assuming that the plastic tension deformation  $\delta_t$  is proportional to the plastic compression deformation  $\delta_c$  as

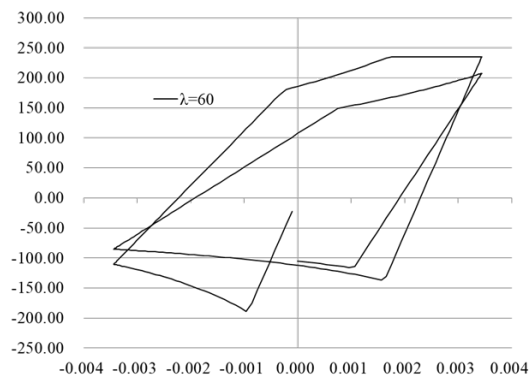
$$\delta_t = q_3 \delta_c$$

where  $q_3 = 0.3\sqrt{n_E} + 0.24$

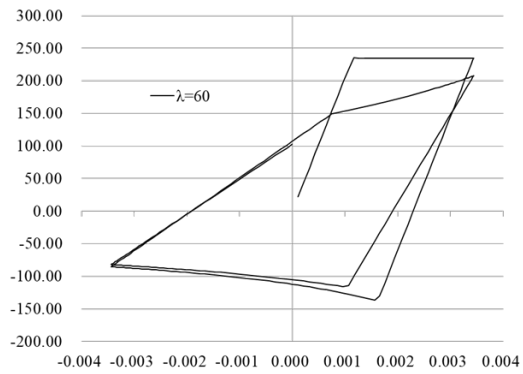


### Example

$\lambda = 60$

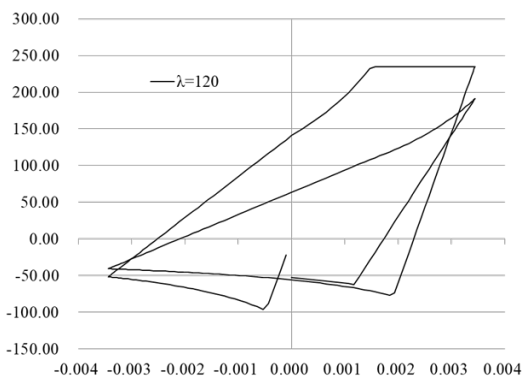


Starting from compression

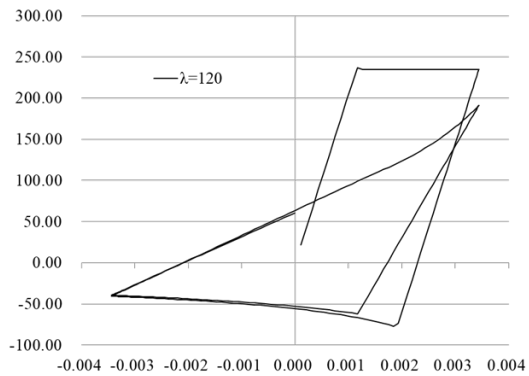


Starting from tension

$\lambda = 120$



Starting from compression



Starting from tension

### References

M. Shibata, T. Nakayama and M. Wakabayashi, "Mathematical Expression of Hysteretic Behavior of Braces", Research Report, Architectural Institute of Japan, No. 316, pp.18-24, 1982.6 (in Japanese)

### 3.3.3 SRC Wall (Brace)

#### a) Section properties

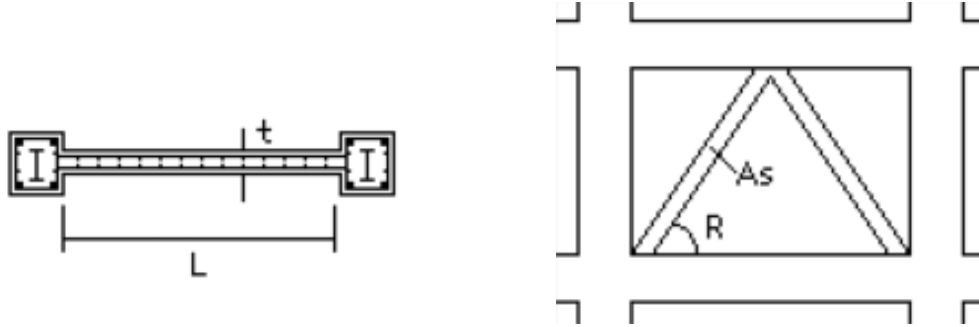


Figure 3-3-17 Element model for SRC wall (RC wall with steel brace)

#### b) Nonlinear shear spring

##### Yield shear force

The yield shear force,  $Q_y$  is calculated as,

$$Q_y = Q_{y,RC} + Q_{y,S} \quad (3-3-34)$$

where

$Q_{y,RC}$  : Yield shear force of reinforced concrete

$$Q_{y,RC} = \left\{ \frac{0.053 p_t^{0.23} (\sigma_B + 18)}{M / (QD) + 0.12} + 0.85 \sqrt{p_w \cdot \sigma_{wy}} + 0.1 \sigma_0 \right\} b \cdot j \quad (3-3-35)$$

$Q_{y,S}$  : Yield shear force of steel

$$Q_{y,S} = A_S \sigma_{y,S} \cos R \quad (3-3-36)$$

where,

$A_S$  : Area of steel ( $\text{mm}^2$ )

$\sigma_{y,S}$  : Strength of steel ( $\text{N/mm}^2$ )

$R$  : Angle of steel

### 3.4 External Spring

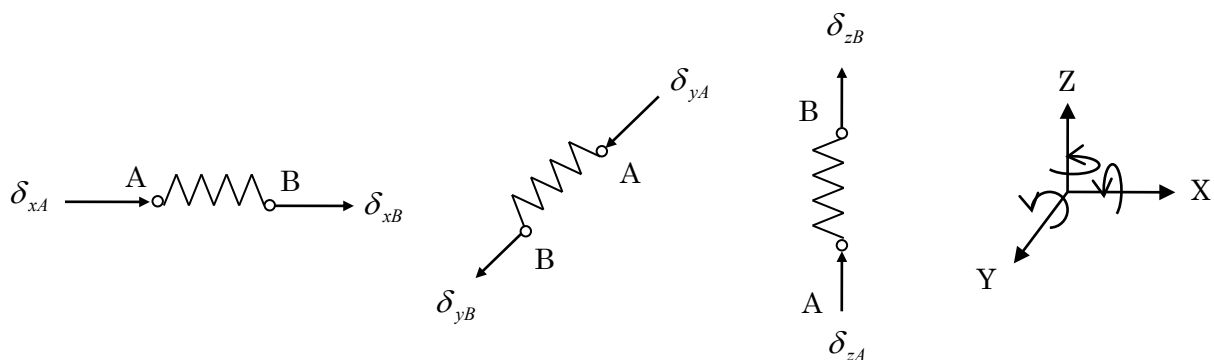


Figure 3-4-1 Element model for external spring

#### 3.4.1 Lift up spring

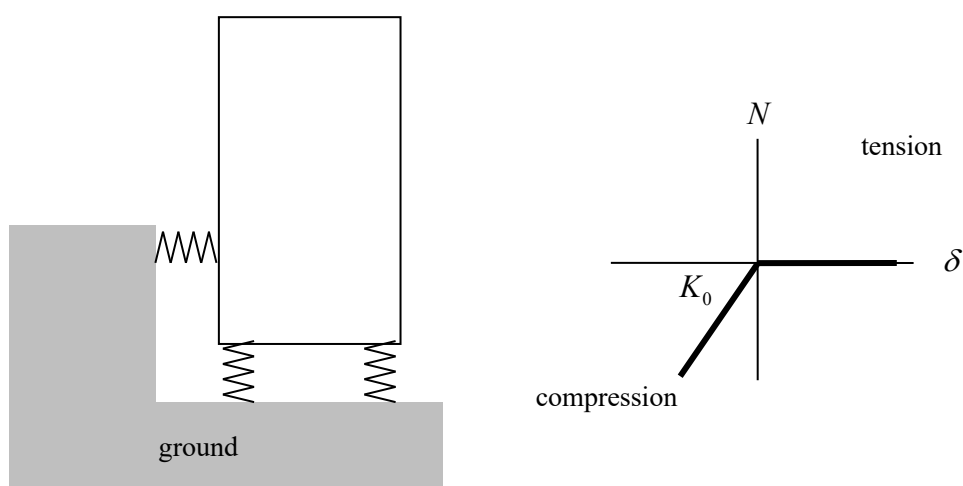


Figure 3-4-2 Hysteresis model of the external spring

In STERA\_3D, if there is no building element at one end of the external spring, this end is considered fixed. Such spring is used to express the stiffness the ground attached to the building. In such a case, as the relationship between axial force and deformation of the spring, the linear stiffness is defined only in compression side and zero stiffness in the tension side as shown in Figure 3-4-2, assuming that the building detaches from the ground.

### 3.4.2 Air spring

Reference:

- 1) Marin Presthus, "Derivation of Air Spring Model Parameters for Train Simulation", Master of Science Programme, Department of Applied Physics and Mechanical Engineering, Luleå University of Technology, Sweden, 2002

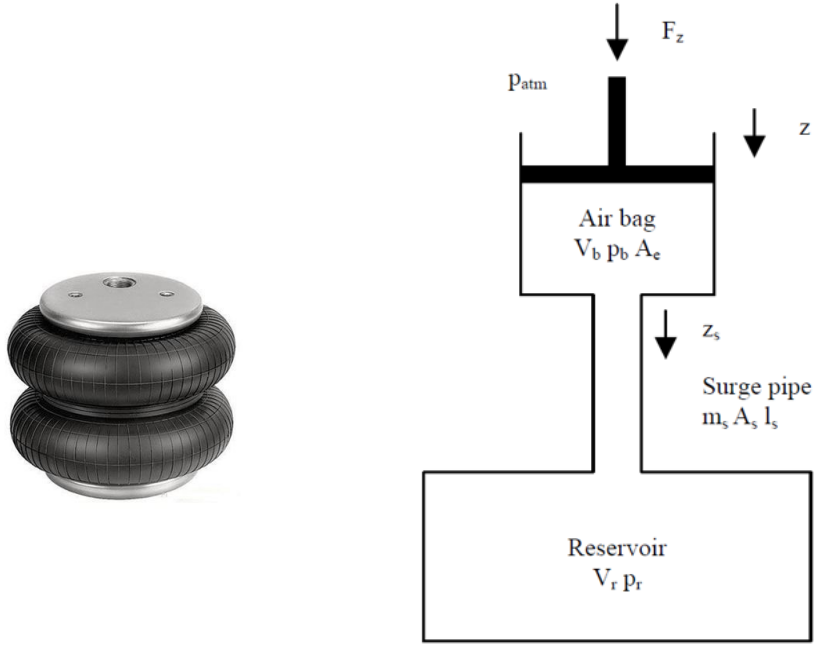


Figure 3-4-3 Air spring ( $V$  : volume,  $p$  : relative pressure,  $A$  : area)

An effective area  $A_e$  is introduced to express the volume change of air bag  $\Delta V_b$  as

$$\Delta V_b = A_e z \quad (3-4-1)$$

When the initial pressure of air spring is  $p_0$ , after the deflection, the pressure will change as

$$p_b = p_0 + \Delta p_b \quad \text{for air bag} \quad (3-4-2a)$$

$$p_r = p_0 + \Delta p_r \quad \text{for reservoir} \quad (3-4-2b)$$

The volume will also change as

$$V_b = V_{b0} - z A_e + z_s A_s \quad \text{for air bag} \quad (3-4-3a)$$

$$V_r = V_{r0} - z_s A_s \quad \text{for reservoir} \quad (3-4-3b)$$

where

$z_s$  : the movement of air mass through orifice

$A_s$  : area of surge pipe

The pressure and the volume of the isentropic process can be described by

$$p_1 \cdot V_1^n = p_2 \cdot V_2^n \quad (3-4-4)$$

where

$p_1, V_1$ : initial pressure and volume

$p_2, V_2$ : final pressure and volume

$n$  : ratio of specific heat = 1.4 for Air

Applying the above equation to the air bag

$$(p_0 + \Delta p_b) \cdot (V_{b0} - zA_e + z_s A_s)^n = p_0 \cdot V_{b0}^n \quad (3-4-5a)$$

$$(p_0 + \Delta p_b) \cdot \left( 1 + \frac{-zA_e + z_s A_s}{V_{b0}} \right)^n = p_0 \quad (3-4-5b)$$

by using Taylor expansion  $(1+x)^n \approx 1+nx$  ( $x \ll 1$ )

$$\left( 1 + \frac{\Delta p_b}{p_0} \right) \cdot \left( 1 + \frac{n(-zA_e + z_s A_s)}{V_{b0}} \right) = 1 \quad (3-4-5c)$$

$$\text{Assuming } \left( \frac{\Delta p_b}{p_0} \right) \left( \frac{-zA_e + z_s A_s}{V_{b0}} \right) \approx 0$$

$$\frac{\Delta p_b}{p_0} \approx \frac{n(zA_e - z_s A_s)}{V_{b0}} \quad (3-4-5d)$$

Using the same procedure for the reservoir

$$(p_0 + \Delta p_r) \cdot (V_{r0} - z_s A_s)^n = p_0 \cdot V_{r0}^n \quad (3-4-6a)$$

$$\frac{\Delta p_r}{p_0} \approx \frac{nz_s A_s}{V_{r0}} \quad (3-4-6b)$$

From the Bernoulli equation, the difference of the pressure between the left and right of the pipe speeds up a portion of gas through the orifice. The force balance in the pipe is given by

$$A_s (\Delta p_b - \Delta p_r) = C_s \dot{z}_s^\beta \quad (3-4-7a)$$

where

$\beta$  : viscous damping parameter determined by experiment

Substituting Eq. (3-4-5d) and (3-4-6b),

$$p_0 A_s n \left( \frac{zA_e - z_s A_s}{V_{b0}} - \frac{z_s A_s}{V_{r0}} \right) = C_s \dot{z}_s^\beta \quad (3-4-7b)$$

$$\frac{np_0 A_s A_e}{V_{b0}} \left( z - \frac{V_{b0} A_s}{A_e} \left( \frac{1}{V_{b0}} + \frac{1}{V_{r0}} \right) z_s \right) = C_s \dot{z}_s^\beta \quad (3-4-7c)$$

The force balance for the piston can be expressed as

$$F_z = A_e (p_b - p_{atm}) \quad (3-4-8)$$

where

$p_{atm}$  : atmospheric pressure

Substituting Eq. (3-4-2a),

$$\begin{aligned} F_z &= (p_0 + \Delta p_b - p_{atm}) A_e \\ &= \Delta p_b A_e + (p_0 - p_{atm}) A_e \\ &= \frac{n(zA_e - z_s A_s)}{V_{b0}} p_0 A_e + (p_0 - p_{atm}) A_e \\ &= \frac{np_0 A_e^2}{V_{b0}} \left( z - \frac{A_s}{A_e} z_s \right) + (p_0 - p_{atm}) A_e \end{aligned} \quad (3-4-9)$$

From Eq. (3-4-7c)

$$\frac{np_0 A_s A_e}{V_{b0}} \left( z - \frac{A_s}{A_e} \left( \frac{V_{b0} + V_{r0}}{V_{r0}} \right) z_s \right) = C_s \dot{z}_s^\beta \quad (3-4-10)$$

$$\begin{aligned} \downarrow \quad \lambda &= \frac{V_{r0}}{V_{b0} + V_{r0}} \\ \frac{np_0 A_e^2}{V_{b0}} \left( z - \frac{A_s}{A_e \lambda} z_s \right) &= \frac{A_e}{A_s} C_s \dot{z}_s^\beta \end{aligned} \quad (3-4-11)$$

$$F_z = \frac{np_0 A_e^2 \lambda}{V_{b0}} \left( \frac{1}{\lambda} z - \frac{A_s}{A_e \lambda} z_s \right) + (p_0 - p_{atm}) A_e \quad (3-4-12)$$

$$K_v = \frac{np_0 A_e^2 \lambda}{V_{b0}} = \frac{np_0 A_e^2}{V_{b0} + V_{r0}} \frac{V_{r0}}{V_{b0}} = K_e \frac{V_{r0}}{V_{b0}}, \quad K_e = \frac{np_0 A_e^2}{V_{b0} + V_{r0}}$$

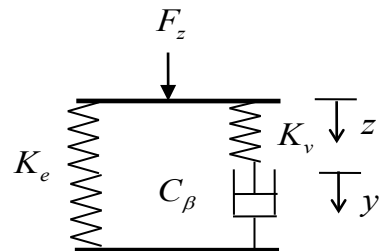
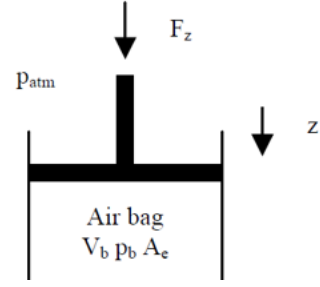
$$\begin{aligned} \downarrow \quad \text{Introducing a new variable } y &= \frac{A_s}{A_e \lambda} z_s \\ K_v (z - y) &= \lambda \frac{A_e}{A_s} \left( \frac{A_e \lambda}{A_s} \right)^\beta C_s \dot{y}^\beta = C_\beta \dot{y}^\beta, \quad C_\beta = \lambda \frac{A_e}{A_s} \left( \frac{A_e \lambda}{A_s} \right)^\beta C_s \end{aligned} \quad (3-4-13)$$

$$F_z = K_v \left( \frac{1}{\lambda} z - y \right) + (p_0 - p_{atm}) A_e = K_v (z - y) + K_v \left( \frac{1}{\lambda} - 1 \right) z + (p_0 - p_{atm}) A_e \quad (3-4-14)$$

Therefore

$$K_v (z - y) = C_\beta \dot{y}^\beta \quad (3-4-15)$$

$$F_z = K_v (z - y) + K_e z + (p_0 - p_{atm}) A_e \quad (3-4-16)$$



Incremental form of equation is

$$F_{z(n+1)} = K_v (z_{(n+1)} - y_{(n+1)}) + K_e z_{(n+1)} \quad (3-4-17)$$

$$z_{(n+1)} = z_{(n)} + \Delta t \cdot \dot{z}(t_{n+1})$$

$$y_{(n+1)} = y_{(n)} + \Delta t \cdot \dot{y}(t_{n+1})$$

Then

$$C_\beta \left( \frac{y_{(n+1)} - y_{(n)}}{\Delta t} \right)^\beta = K_v (z_{(n+1)} - y_{(n+1)}) \quad (3-4-18)$$

The solution of Eq. (3-4-18) is obtained by solving the following equation:

$$f(y_{(n+1)}) = C_\beta \left( \frac{y_{(n+1)} - y_{(n)}}{\Delta t} \right)^\beta - K_v (z_{(n+1)} - y_{(n+1)}) = 0 \quad (3-4-19)$$

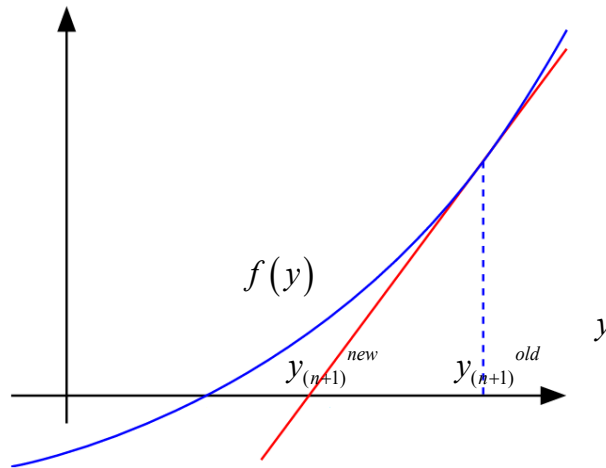
Its derivative regarding  $y_{(n+1)}$  is

$$f'(y_{(n+1)}) = \frac{\beta C_\beta}{\Delta t} \left( \frac{y_{(n+1)} - y_{(n)}}{\Delta t} \right)^{\beta-1} + K_v \quad (3-4-20)$$

A Newton-Raphson method is applied to solve the nonlinear equation  $f(y_{(n+1)}) = 0$

$$y_{(n+1)}^{new} = y_{(n+1)}^{old} - \frac{f(y_{(n+1)})}{f'(y_{(n+1)})} \quad (3-4-21)$$

where the prime  $f'(y_{(n+1)})$  denotes derivative with respect to  $y_{(n+1)}$ ,



### 3.4.3 Base plate

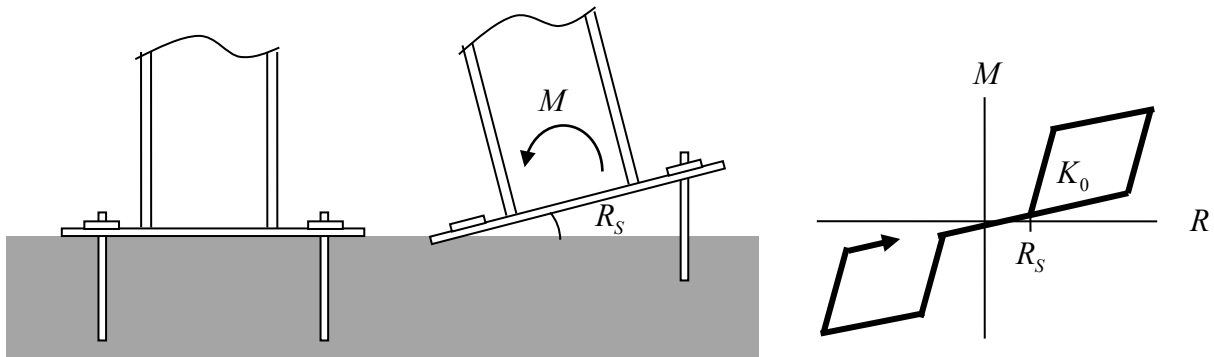


Figure 3-4-4 Hysteresis model of the base plate

The relationship between the moment at the bottom of the steel column,  $M$ , and the rotation angle of the base plate,  $R$ , is given as a rotational spring. The hysteresis model of the  $M$ - $R$  relationship takes into account plastic deformation due to anchor bolt pullout and tensile yielding.

### 3.4.4 Pendulum element

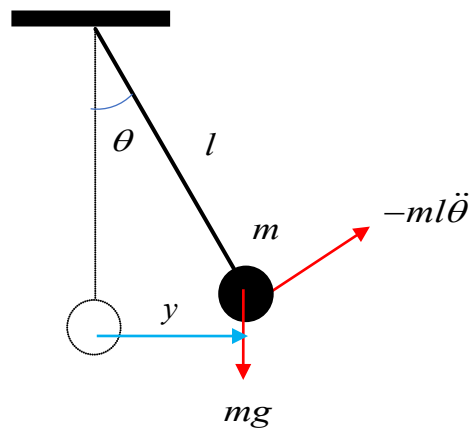


Figure 3-4-5 Pendulum element

From the equilibrium condition of the moment force

$$ml^2\ddot{\theta} + mgl \sin \theta = 0 \quad (3-4-22)$$

Setting  $y \approx l\theta$ ,  $\sin \theta \approx \theta$

$$m\ddot{y} + mgl\theta = 0 \rightarrow m\ddot{y} + mgy = 0 \rightarrow m\ddot{y} + \left(\frac{mg}{l}\right)y = 0 \rightarrow \ddot{y} + \left(\frac{g}{l}\right)y = 0 \quad (3-4-23)$$

Therefore, the natural period of the pendulum element is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

It means that the pendulum element is equivalent to the element with the horizontal stiffness,  $k_h = \frac{mg}{l}$ .

$$m\ddot{y} + \left(\frac{mg}{l}\right)y = 0 \rightarrow m\ddot{y} + k_h y = 0, \quad k_h = \frac{mg}{l}$$

Furthermore, the horizontal stiffness of the member with the initial tensile force,  $T$ , can be interpreted as  $k_h = \frac{T}{l}$ , even in the static condition.

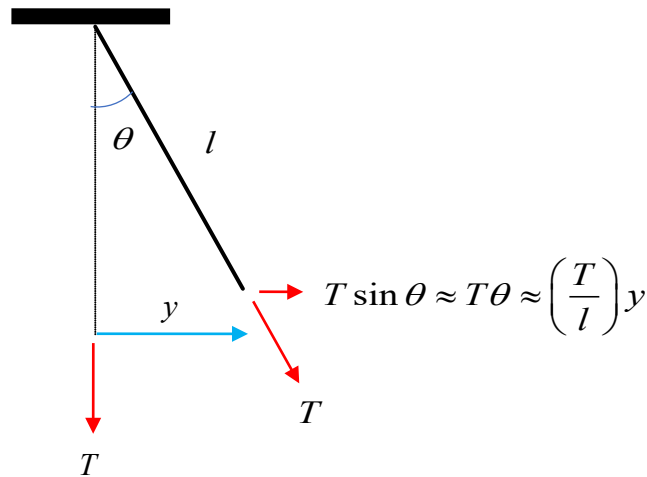


Figure 3-4-5 Pendulum element

Therefore, the pendulum element can be interpreted as a line element with the axial stiffness,  $k_v$ , and the

horizontal stiffness,  $k_h = \frac{T}{l}$ , where  $T$  is calculated by the gravity force.

### 3.5 Base Isolation

The element model of base isolation consists of shear springs arranged in x-y plane changing its direction with equal angle interval as shown in Figure 3-5-1. This model is called MSS (Multi-Shear Spring) model developed by Wada et al.

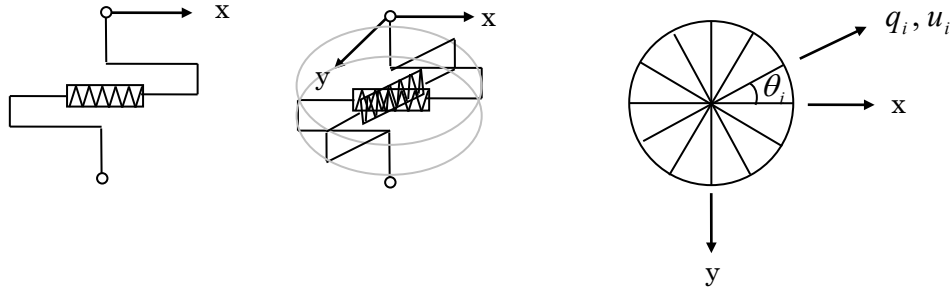


Figure 3-5-1 Element model of base isolation

#### a) Nonlinear shear spring

The hysteresis model of each nonlinear shear spring is defined as a bi-linear model as shown in Figure 3-5-2. The force and displacement vectors of i-th shear spring are expressed as,

$$\begin{Bmatrix} q_{i,x} \\ q_{i,y} \end{Bmatrix} = \begin{bmatrix} \cos \theta_i \\ \sin \theta_i \end{bmatrix} q_i \quad (3-5-1)$$

$$u_i = \begin{bmatrix} \cos \theta_i & \sin \theta_i \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \quad (3-5-2)$$

From the relationship,  $q_i = k_i u_i$ , the constitutive equation of i-th shear spring is,

$$\begin{Bmatrix} q_{i,x} \\ q_{i,y} \end{Bmatrix} = k_i \begin{bmatrix} \cos \theta_i \\ \sin \theta_i \end{bmatrix} \begin{bmatrix} \cos \theta_i & \sin \theta_i \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ \cos \theta_i \sin \theta_i & \sin^2 \theta_i \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \quad (3-5-3)$$

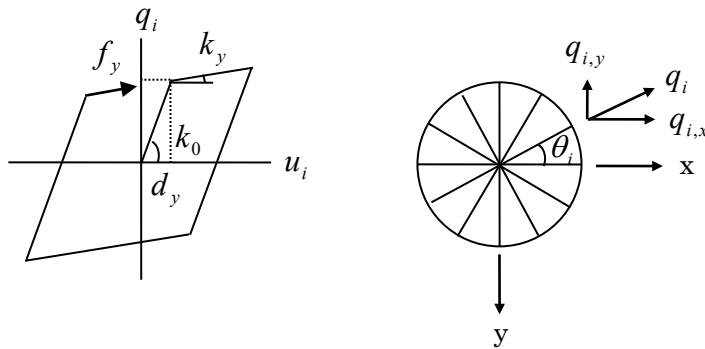


Figure 3-5-2 Hysteresis model of the shear spring

From the sum of all nonlinear shear springs in the element, the constitutive equation of the base isolation element is,

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \left( \sum_{i=1}^N k_i \begin{bmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ \cos \theta_i \sin \theta_i & \sin^2 \theta_i \end{bmatrix} \right) \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \quad (3-5-4)$$

where,  $N$  is the number of shear springs in an element. In STERA\_3D,  $N=6$  is selected.

### First and second stiffness

We assume that all nonlinear shear springs in an element have the same stiffness and strength. The initial stiffness of the base isolation element,  $K_0$ , is obtained from Equation (3-5-4) by substituting  $u_x = 1, u_y = 0$ .

$$K_0 = \left( \sum_{i=1}^N \cos^2 \theta_i \right) k_0 \quad (3-5-5)$$

Therefore, the initial stiffness of each shear spring is,

$$k_0 = \frac{K_0}{\sum_{i=1}^N \cos^2 \theta_i} \quad (3-5-6)$$

The same relationship is established for the second stiffness after yielding,

$$k_y = \frac{K_y}{\sum_{i=1}^N \cos^2 \theta_i} \quad (3-5-7)$$

where,  $K_y$  and  $k_y$  are the second stiffness after yielding for the base isolation element and the nonlinear shear spring, respectively.

### Yield shear force

The yield shear force of the base isolation element,  $Q_y$ , is obtained assuming that all the nonlinear shear springs reach their yielding points except the spring perpendicular to the loading direction, and the increase of the force after yielding is negligible (Figure 3-5-3). That is,

$$Q_y = \left( \sum_{i=1}^N |\cos \theta_i| \right) f_y \quad (3-5-8)$$

Therefore, the yield shear force of each shear spring is,

$$f_y = \frac{Q_y}{\sum_{i=1}^N |\cos \theta_i|} \quad (3-5-9)$$

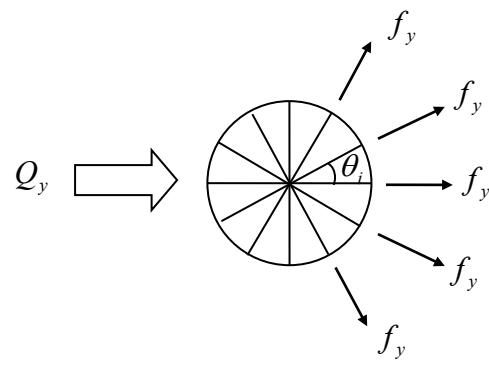


Figure 3-5-3 Assumption of yield shear force

### Appendix 3.5:

#### A-1. Hysteresis of LRB (Lead Rubber Bearing)

LRB (Lead Rubber Bearing) is composed by rubber layers, steel plates and a lead plug core.

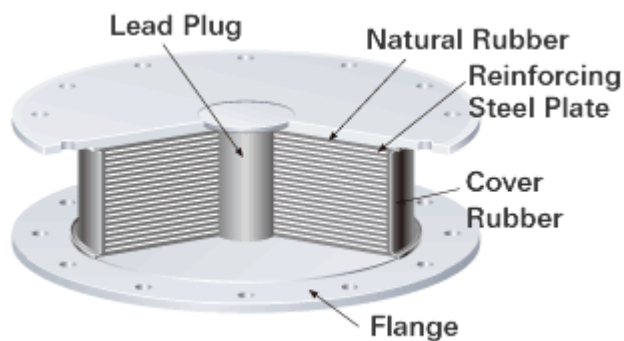


Figure A1-1. Lead Rubber Bearing (from Bridgestone Catalog)

##### 1) Bi-Linear Model

The bi-linear hysteresis of LRB is defined as a combination of an elastic model and elasto-plastic model as shown Figure A1-2.

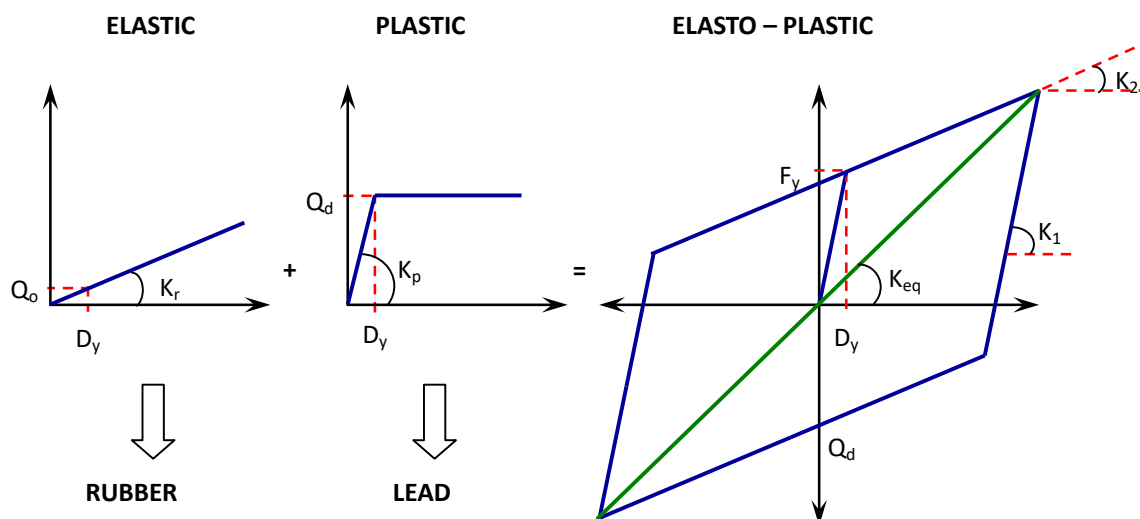


Figure A1-2. Bi-linear model

The elastic stiffness,  $K_r$ , from the rubber is calculated as,

$$K_r = G_r \frac{A_r}{H_r} \quad (A1-1)$$

where  $G_r$  is the shear modulus of the rubber,  $A_r$  is the cross section area of the rubber and  $H_r$  is the total height of the rubber.

The elastic stiffness,  $K_p$ , from the lead plug is calculated as,

$$K_p = G_p \frac{A_p}{H_p} \quad (\text{A1-2})$$

where  $G_p$  is the shear modulus of lead,  $A_p$  is the cross section area of lead plug and  $H_p$  is the total height of the plug.

The initial elastic stiffness,  $K_1$ , and the secondary stiffness,  $K_2$ , of the bi-linear model are then obtained as,

$$\begin{aligned} K_1 &= K_r + K_p \\ K_2 &= K_r \end{aligned} \quad (\text{A1-3})$$

The yielding deformation,  $D_y$ , is determined from the characteristics of the lead plug. The yielding force,  $F_y$ , is calculated as,

$$F_y = (K_r + K_p) D_y \quad (\text{A1-4})$$

## 2) Modified Bi-linear Model

Hysteresis of a lead rubber bearing has a characteristic of stiffness degrading according to the strain level as shown in Figure A1-3.

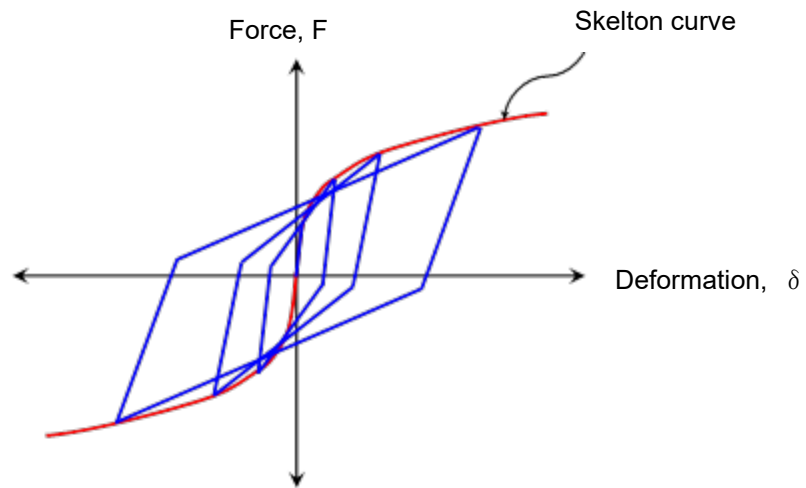


Figure A1-3. Hysteresis of a lead rubber bearing

The secondary stiffness of a lead rubber bearing,  $K_d$ , is expressed as,

$$K_d(\gamma) = C_{Kd}(\gamma)(K_r + K_p) \quad (\text{A1-5})$$

where  $\gamma$  is a strain ratio ( $\gamma = \delta / H_r$ ) and  $C_{Kd}(\gamma)$  is a modification factor of the secondary stiffness, which takes into consideration the strain dependency. Also, the intercept force is defined as,

$$Q_d(\gamma) = C_{Qd}(\gamma) \sigma_p A_p \quad (\text{A1-6})$$

where  $C_{Qd}(\gamma)$  is a modification factor of the yielding shear force and  $\sigma_p$  is the yielding shear stress of lead.

The force is then expressed by:

$$F(\gamma) = K_d(\gamma) \delta + Q_d(\gamma) \quad (\text{A1-7})$$

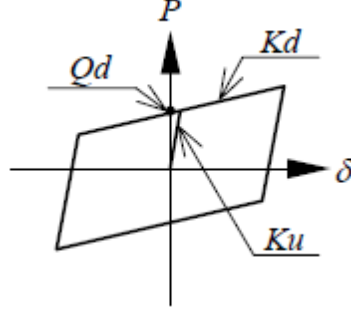


Figure A1-4 Hysteresis loop model of lead rubber bearing

The modification factors,  $C_{Kd}(\gamma)$  and  $C_{Qd}(\gamma)$ , are represented by the following formulas under 15 degrees Celsius.

$$C_{Kd}(\gamma) = \begin{cases} 0.779\gamma^{-0.43}, & \gamma < 0.25 \\ \gamma^{-0.25}, & 0.25 \leq \gamma < 1.0 \\ \gamma^{-0.12}, & 1.0 \leq \gamma < 2.5 \end{cases} \quad (\text{A1-8})$$

$$C_{Qd}(\gamma) = \begin{cases} 2.036\gamma^{0.41}, & \gamma < 0.1 \\ 1.106\gamma^{0.145}, & 0.1 \leq \gamma < 0.5 \\ 1, & 0.5 \leq \gamma \end{cases} \quad (\text{A1-9})$$

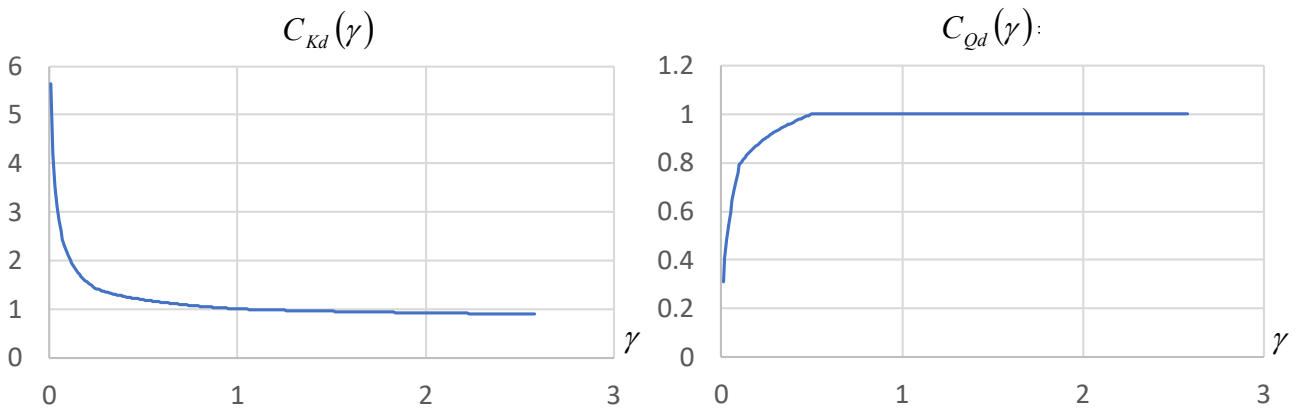


Figure A1-5. Modification factors

Under a different temperature,  $t$ , the secondary stiffness and the yielding shear force are to be corrected by the following formulas:

$$K_d(t) = K_d(t_0) \exp(-0.00271(t - t_0)) \quad (\text{A1-10})$$

$$Q_d(t) = Q_d(t_0) \exp(-0.00879(t - t_0)) \quad (\text{A1-11})$$

where  $t_0 = 15$  degree Celsius.

The primary stiffness of the lead rubber bearing,  $K_u$ , in Figure A1-4 is determined from the secondary stiffness,  $K_d$ , as,

$$K_u = \beta K_d \quad (\text{A1-12})$$

where  $10 \leq \beta \leq 15$ .

Following the suggestion in the manual of CANNY (K. Li, 2004), the hysteresis rules are:

a) Elastic range

Under the strain level less than  $\gamma_e$ , the hysteresis is assumed to be linear with the secant stiffness at the strain,  $\gamma_e$ , that is:

$$K_0 = F_e / \gamma_e \quad (\text{A1-13})$$

$$F_e = K_d(\gamma_e) \delta_e + Q_d(\gamma_e), \quad \delta_e = \gamma_e H_r \quad (\text{A1-14})$$

The value,  $\gamma_e = 0.01$ , is adopted in STERA3D.

b) Loading on the skeleton curve after elastic range

Under the loading on the skeleton curve after elastic range, tangent stiffness is used to estimate the response at the next step:

$$K = dF(\gamma) / d\gamma \quad (\text{A1-15})$$

**Reference:**

*Response Control and Seismic Isolation of Buildings*, Edited by Masahiko Higashino and Shin Okamoto, SPON PRESS, October 17, 2006.

*Canny Technical Manual*, Kangning Li, August 2004

## 2) Consideration of strength reduction by dissipated energy

### Reference

- 1) Masanori Iiba, et.al., “Research on Characteristics of Isolators and Dampers under Multi-cyclic Earthquake Motions and Effects on Response of Seismically Isolated Buildings”, Building Research Institute, National Research and Development Agency, Building Research Data, No. 170, April 2016 (in Japanese).
- 2) Haruyuki Kitamura and Miyuki Omiya, “Design method for long period ground motion - Points to note when dealing with long-period ground motion”, The Kenchiku Gijyutsu, No. 815, pp.116-125, 2017.12 (in Japanese)

From Reference 1), the yield shear stress of lead plug,  $\tau$ , is expressed as

$$\tau = \tau_0 \left\{ 1 - (T/T_L)^{\alpha_T} \right\}, \quad \alpha_T = 0.4 + 0.25(T/T_L) \quad (\text{A1-16})$$

Where,

$\tau_0$  : Design value of the yield shear stress of lead plug = 15.0 (N/mm<sup>2</sup>)

$T$  : Average temperature of lead plug

$T_L$  : Melting point of lead plug = 327.5 (°C)

For example, when  $T=20$  (°C),  $\tau$  is calculated to be 10.3 (N/mm<sup>2</sup>).

Reference 2) suggested another formula as

$$Q_d'(\gamma) = \mu Q_d(\gamma) \quad (\text{A1-17})$$

$$\mu = -0.06 + 1.25 \exp \left( -\frac{1}{360} \frac{W_{pb}}{V_{pb}} \right) \quad (\text{A1-18})$$

where

$Q_d(\gamma)$  : Intercept force without reduction

$\mu$  : Reduction factor

$W_{pb}$  : Dissipated energy

$V_{pb} = \frac{\pi}{4} D_{pb}^2 h_{pb}$  : Volume of lead plug

$h_{pb} = nt_r + (n-1)t_s$  : Height of lead plug

$n$  : number of rubber layer,  $t_r$  : thickness of rubber layer,  $t_s$  : thickness of steel plate

Also, the following formula is sometimes used

$$\mu = \frac{8.33}{7.97} \left\{ -0.06 + 1.25 \exp \left( -f(D_{pb}) \frac{1}{360} \frac{W_{pb}}{V_{pb}} \right) \right\} \quad (\text{A1-19})$$

where

$f(D_{pb}) = 0.16 D_{pb}^{0.31}$  : Correction value by the diameter of the lead plug,  $D_{pb}$  (mm)

STERA\_3D adopts Equation (A1-19). The reduction factor  $\mu$  is plotted as a function of energy dissipation as follows.

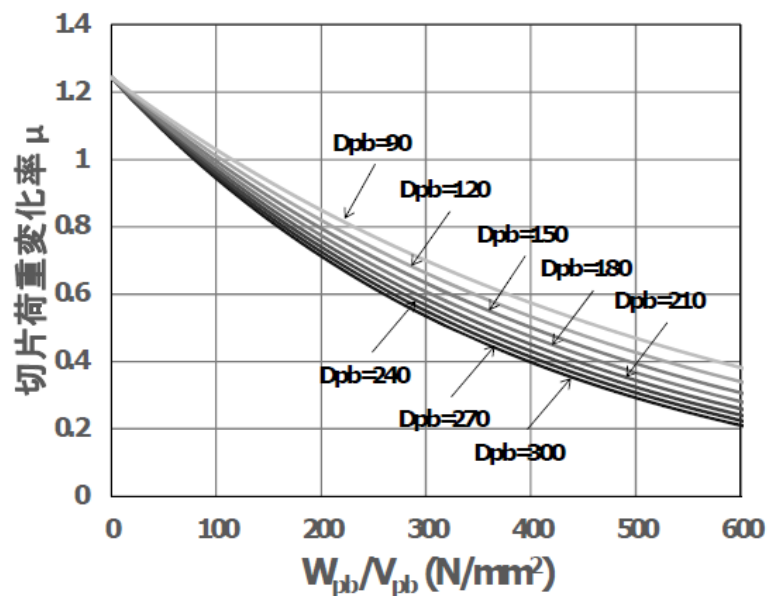


Figure A1-6. Strength reduction factor by energy dissipation

Example)

Bridgestone Product: LH060G4 C

Diameter (mm)	600
Lead plug diameter (mm)	100
Effective area ( $\times 10^2 \text{mm}^2$ )	2749
Thickness of one rubber layer (mm)	4
Number of rubber layers	50
Total rubber thickness (mm)	200
Total height (mm)	407.9
Shear modulus of rubber $G_r$ ( $\text{N/mm}^2$ )	0.385
Apparent shear modulus of lead $\alpha_p$ ( $\text{N/mm}^2$ )	0.583
Yield shear stress of lead $s_y$ ( $\text{N/mm}^2$ )	7.967
(shear properties at shear strain = 100%)	
Initial stiffness $K_1$ ( $\times 10^3 \text{kN/m}$ )	7.18 (=13 $\times K_2$ )
Post yield stiffness $K_2$ ( $\times 10^3 \text{kN/m}$ )	0.552 <sup>*1)</sup>
Characteristic strength $Q_d$ (kN)	62.6 <sup>*1)</sup>

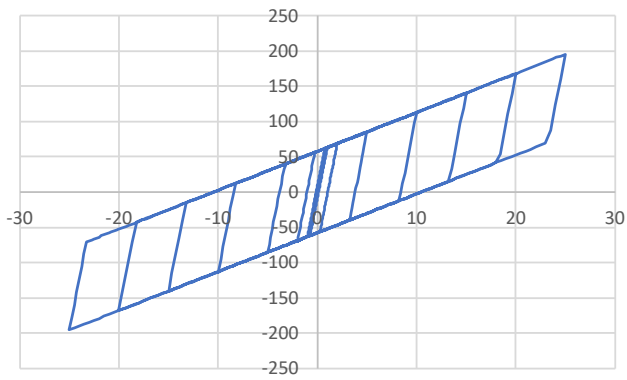
\*1)

Shear stiffness of laminated rubber  $K_r = G_r A_r / H$  ( $\times 10^3 \text{kN/m}$ ) 0.529

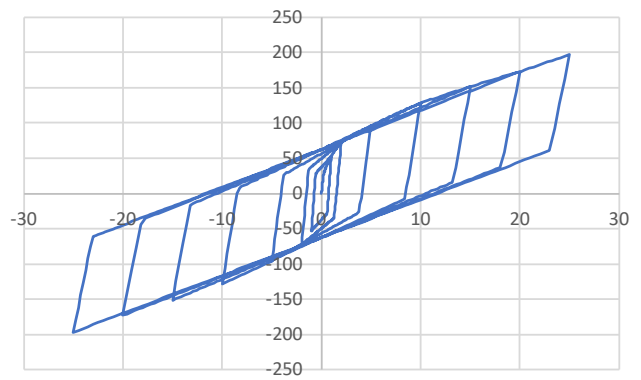
Additional shear stiffness by lead plug :  $K_p = \alpha_p A_p / H$  ( $\times 10^3 \text{kN/m}$ ) 0.023

Total stiffness  $K_2 = K_r + K_p$  ( $\times 10^3 \text{kN/m}$ ) 0.552

Yield strength of lead  $Q_d = s_y A_p$  (kN) 62.573



(a) Bi-linear



(b) Modified bi-linear

Figure A1-7. Comparison of hysteresis loops

## A-2. Hysteresis of HDRB (High Damping Rubber Bearing)

HDRB (High Damping Rubber Bearing) is composed by rubber layers and steel plates. By adding special ingredient in the natural rubber, rubber itself demonstrates damping characteristics.

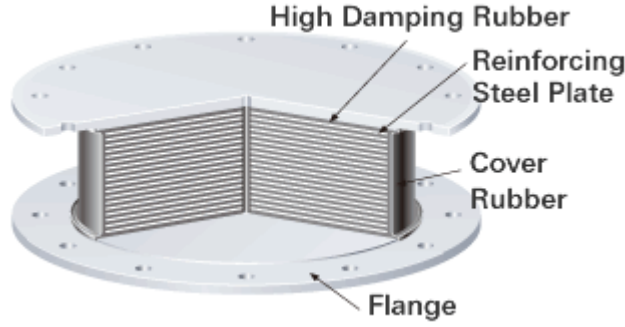


Figure A2-1. High Damping Rubber Bearing (from Bridgestone Catalog)

### 1) Modified Bi-linear Model

The hysteresis of HRB is defined as a modified bilinear model as shown Figure A2-2.

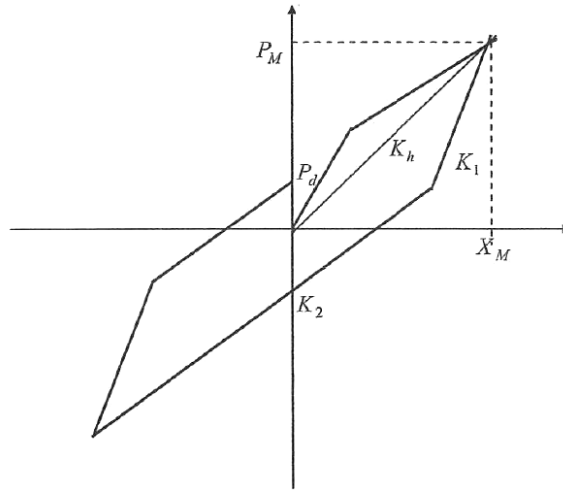


Figure A2-2. Bi-linear model

The initial stiffness,  $K_1$ , from the rubber is calculated as,

$$K_1(\gamma) = G_1(\gamma) \times A / H_r$$

$$G_1(\gamma) = \frac{u(\gamma) - \pi \times h_{eq}(\gamma) / 2 + u(\gamma) \times \pi \times h_{eq}(\gamma) / 2}{u(\gamma) - \pi \times h_{eq}(\gamma) / 2} \times G_{eq}(\gamma) \quad (A2-1)$$

where  $\gamma$  : shear strain ( $\gamma = \delta / H_r$ )

$A_r$  : cross section area of the rubber

$H_r$  : total height of the rubber.

$G_{eq}(\gamma)$  : Equivalent shear modulus

$$G_{eq}(\gamma) = \alpha_0 + \alpha_1\gamma + \alpha_2\gamma^2 + \alpha_3\gamma^3 + \cdots + \alpha_n\gamma^n \quad (A2-2)$$

$h_{eq}(\gamma)$  : Equivalent damping factor

$$h_{eq}(\gamma) = \beta_0 + \beta_1\gamma + \beta_2\gamma^2 + \beta_3\gamma^3 + \cdots + \beta_n\gamma^n \quad (A2-3)$$

$u(\gamma)$  : Intercept force

$$u(\gamma) = \mu_0 + \mu_1\gamma + \mu_2\gamma^2 + \mu_3\gamma^3 + \cdots + \mu_n\gamma^n \quad (A2-4)$$

Example)

Diameter:  $\phi 1500$

Thickness of rubber:  $7.5mm \times 20 layers = 150mm$

$S_1 = 49.7$

$S_2 = 10.0$

Nominal compression stress:  $10N / mm^2$

	Strain $\gamma$	Coefficient of each order				
		0	1st	2nd	3rd	4th
$G_{eq}(\gamma)$ ( $N / mm^2$ )	0.1 ~ 1.5	1.1503	-2.5382	3.3047	-2.0356	0.4728
	1.5 ~ 2.5	3.7412	-6.8745	5.1256	-1.6946	0.2092
	2.5 ~ 3.0	0.1749	0.0261	0.0071		
$h_{eq}(\gamma)$	0.1 ~ 1.5	0.135	0.0903	-0.13	0.1067	-0.032
	1.5 ~ 2.5	-0.6239	1.5853	-1.1493	0.3627	-0.0427
	2.5 ~ 3.0	-0.05016	0.1762	-0.0376		
$u(\gamma)$	0.1 ~ 1.5	0.2989				
	1.5 ~ 3.0	0.3685	-0.0464			

Hysteresis of a high damping rubber bearing has a characteristic of stiffness degrading according to the strain level as shown in Figure A2-3.

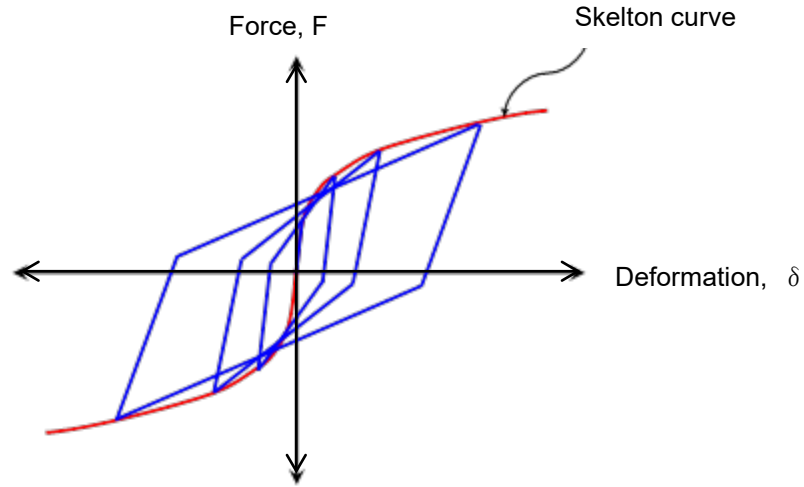


Figure A2-3. Hysteresis of a high damping rubber bearing

The secondary stiffness of a lead rubber bearing,  $K_2$ , is expressed as,

$$K_2(\gamma) = G_2(\gamma) \times A / H_r \quad (\text{A2-5})$$

$$G_2(\gamma) = (1 - u) \times G_{eq}(\gamma) \quad (\text{A2-6})$$

The shear force is defined as,

$$P_M(\gamma) = K_h(\gamma) \times X_M \quad (\text{A2-7})$$

$$K_h(\gamma) = G_{eq}(\gamma) \times A / H_r \quad (\text{A2-8})$$

where  $X_M$ : the maximum deformation

Also intercept force is defined as,

$$P_d(\gamma) = u(\gamma) \times P_M(\gamma) \quad (\text{A2-9})$$

The hysteresis rules are:

a) Elastic range

Under the strain level less than  $\gamma = 0.01$ , the hysteresis is assumed to be linear with the secant stiffness at the strain, that is:

$$K_0 = K_h(\gamma = 0.01) \quad (\text{A2-10})$$

b) Loading on the skeleton curve after elastic range

Under the loading on the skeleton curve after elastic range, tangent stiffness is used to estimate the response at the next step:

$$K = dQ(\gamma) / d\gamma \quad (\text{A2-11})$$

## 2) Consideration of strength reduction by dissipated energy

### Reference

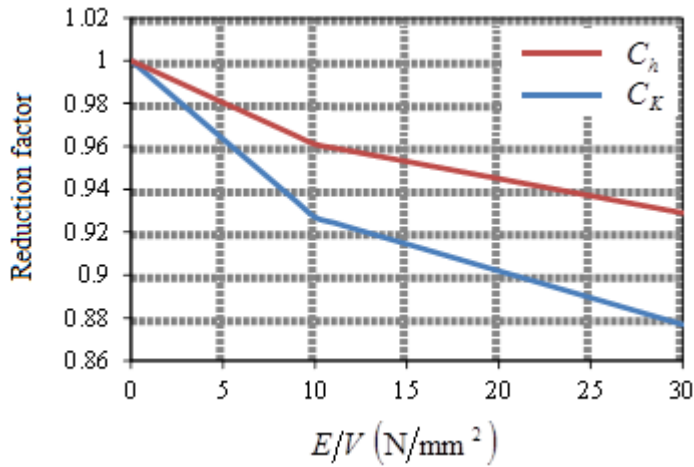
- 3) Takuya Nishimura et al., “Practical Response Evaluation Method for Seismic Isolation System against Long Period Earthquake Motions - Part2- High-Damping Rubber Bearing and Lead Damper”, AIJ Annual Convention, Architectural Institute of Japan, 2013, pp.767-768 (in Japanese)

In the above reference, the reduction factors of equivalent stiffness and equivalent damping are proposed as,

$$\begin{aligned} C_K &= -0.0073 \cdot (E/V) + 1.0 \quad (E/V \leq 10.0 \text{ N/mm}^2) \\ C_K &= -0.0025 \cdot (E/V) + 0.952 \quad (E/V > 10.0 \text{ N/mm}^2) \end{aligned} \quad (\text{A2-12})$$

$$\begin{aligned} C_h &= -0.0039 \cdot (E/V) + 1.0 \quad (E/V \leq 10.0 \text{ N/mm}^2) \\ C_h &= -0.0016 \cdot (E/V) + 0.977 \quad (E/V > 10.0 \text{ N/mm}^2) \end{aligned} \quad (\text{A2-13})$$

where  $E$ : dissipated energy,  $V$ : volume of rubber



To consider the strength reduction by energy dissipation, STERA\_3D modifies the equivalent shear modulus and the equivalent damping factor as,

$$G_{eq}(\gamma) = C_K G_{eq}(\gamma = 0.01) \quad (\text{A2-14})$$

$$h_{eq}(\gamma) = C_h h_{eq}(\gamma = 0.01) \quad (\text{A2-15})$$

### A-3. Hysteresis of Lead Damper

#### Reference

- 1) Takuya Nishimura et al., “Experimental Study on Multi-cyclic Characteristics of Devices for Seismic Isolation against Long Period Earthquake Motions: Part 7- Lead Damper”, AIJ Annual Convention, Architectural Institute of Japan, 2011, pp.667-668 (in Japanese)
- 2) Takuya Nishimura et al., “Study on Multi-cyclic Modeling of Devices and Response Evaluation for Seismic Isolation against Long Period Earthquake Motions: Part 5-Modeling of Lead Damper and Seismic Response Analyses”, AIJ Annual Convention, Architectural Institute of Japan, 2012.9, pp.383-384 (in Japanese)

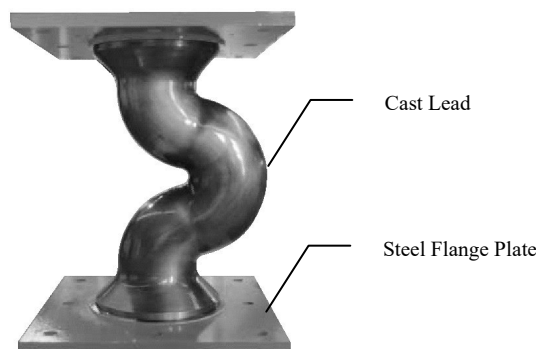


Figure A3-1. Lead damper

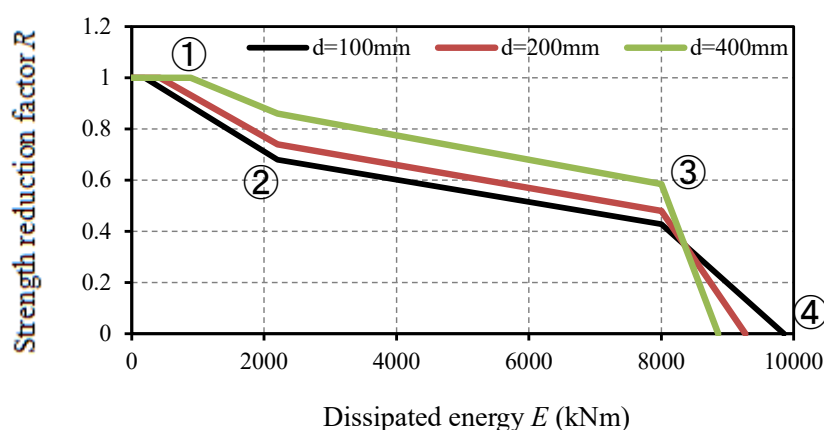


Figure A3-2. Relationship between dissipated energy and strength reduction factor

In the above references, from the cyclic loading test of a lead damper with the different horizontal displacement amplitudes, three line graphs are obtained for the relationship between the dissipated hysteresis energy and the horizontal strength reduction ratio. The breaking points of the line are proposed as follows to match the test results.

a) The first point of strength reduction,  $(R_1, E_1)$

$$R_1 = 1.0$$

$$E_1 = -37 + 2322|d| \quad (0.1 \leq |d| \leq 0.4), \quad 195 \quad (|d| < 0.1), \quad 892 \quad (|d| > 0.4) \quad (\text{kNm}) \quad (\text{A3-1})$$

b) The second point of strength reduction,  $(R_2, E_2)$

$$R_2 = 0.62 + 0.60|d| \quad (0.1 \leq |d| \leq 0.4), \quad 0.680 \quad (|d| < 0.1), \quad 0.860 \quad (|d| > 0.4)$$

$$E_2 = 2,205 \quad (\text{kNm}) \quad (\text{A3-2})$$

c) The third point of strength reduction,  $(R_3, E_3)$

$$R_3 = 0.375 + 0.525|d| \quad (0.1 \leq |d| \leq 0.4), \quad 0.428 \quad (|d| < 0.1), \quad 0.585 \quad (|d| > 0.4)$$

$$E_3 = 8,000 \quad (\text{kNm}) \quad (\text{A3-3})$$

d) The fourth point of strength reduction,  $(R_4, E_4)$

$$R_4 = 0$$

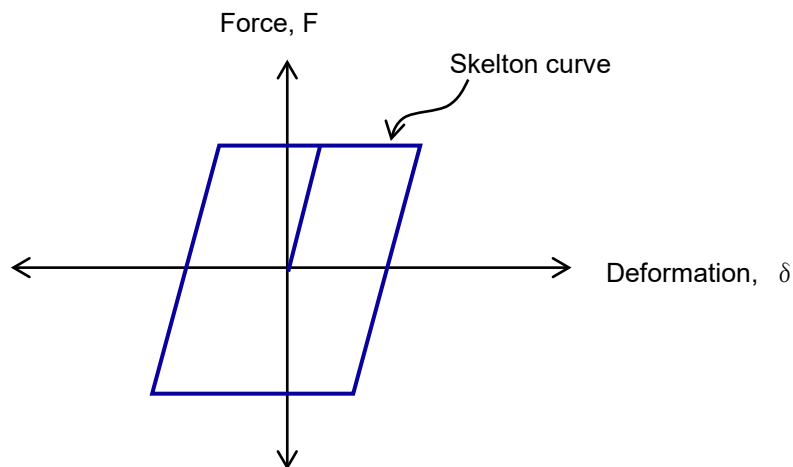
$$E_4 = 9683 - 2060|d| \quad (0.15 \leq |d| \leq 0.4), \quad 9854 \quad (|d| < 0.15), \quad 8859 \quad (|d| > 0.4) \quad (\text{kNm}) \quad (\text{A3-4})$$

The hysteresis of the lead damper is defined as a bilinear model. To consider the strength reduction by energy dissipation, STERA\_3D adopts the line of  $d = 0.2$  (m) for random amplitude. The strength of a lead damper,  $Q_d$ , is then expressed as,

$$Q_d = R Q_{d0} \quad (\text{A3-5})$$

where,  $R$  : Strength reduction factor

$Q_{d0}$  : Initial strength of a lead damper



#### A-4. Hysteresis of Elastic Sliding Bearing

##### Reference

- 1) Shigeo Minewaki et al., “Study on Multi-cyclic Modeling of Devices and Response Evaluation for Seismic Isolation against Long Period Earthquake Motions : Part 2-Modeling of Low Friction Bearing and Viscous Damper”, AIJ Annual Convention, Architectural Institute of Japan, 2012, pp.377-378 (in Japanese)

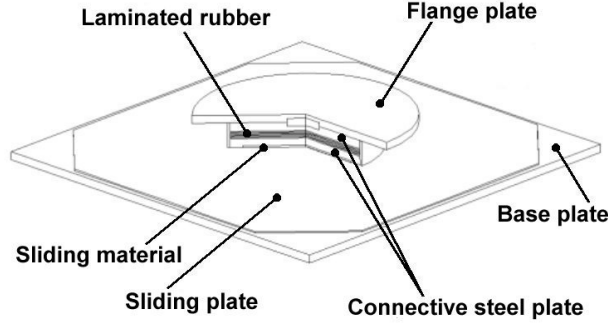


Figure A4-1. Elastic Sliding Bearing

In the above reference, the dynamic friction coefficient changes according to the temperature of the sliding plate as,

$$\mu_0 = -7.5 \times 10^{-5} \cdot T + 0.0145 \quad (\text{A4-1})$$

The change of the friction coefficient is expressed as a function of the increment of temperature as

$$\Delta\mu = 0.03 \cdot (\Delta T + 1)^{-0.06} - 0.03 \quad (\text{A4-2})$$

On the other hand, the increment of temperature has the following relationship with the dissipated energy

$E_d$  (kNmm),

$$\Delta T = 0.00019 \cdot E_d^{0.9} \quad (\text{A4-3})$$

Therefore, the dynamic friction coefficient is obtained from the dissipated energy,

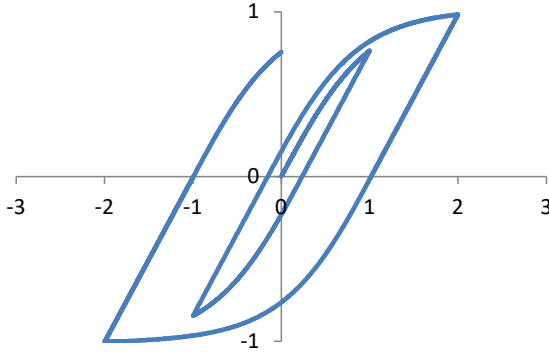
$$\mu = \mu_0(T) + \Delta\mu(E_d) \quad (\text{A4-4})$$

The hysteresis of the elastic sliding bearing is defined as a bilinear model. In STERA\_3D, the initial friction coefficient is temporary assumed as  $\mu_0 = 0.029$  from the catalog of a manufacture. The strength reduction by energy dissipation will be expressed as,

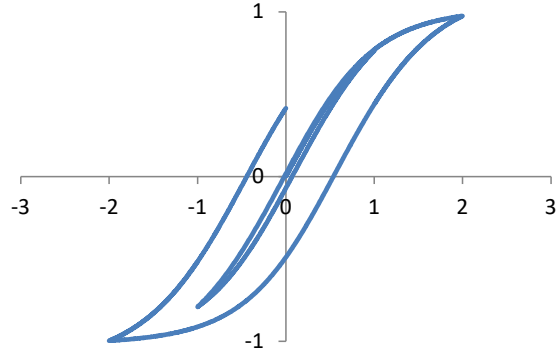
$$Q_d = (\mu_0 + \Delta\mu(E_d)) (Q_{d0} / \mu_0) \quad (\text{A4-5})$$

where,  $Q_{d0}$ : Initial strength of an elastic sliding bearing

### A-5. Hysteresis of Bouc-Wen Model



$$\beta=0.5, \gamma=0.5$$



$$\beta=0.1, \gamma=0.9$$

#### Reference

- 1) Terje Haukaas and Armen Der Kiureghian, "Finite Element Reliability and Sensitivity Methods for Performance-Based Earthquake Engineering", PEER 2003/14, APRIL 2004
- 2) Wen, Y.-K. (1976) "Method for random vibration of hysteretic systems." Journal of Engineering Mechanics Division, 102(EM2), 249-263.
- 3) Baber, T. T. and Noori, M. N. (1985). "Random vibration of degrading, pinching systems." Journal of Engineering Mechanics, 111(8), 1010-1026.

#### a) Basic formulation

The basic formula of Bouc-Wen model is

$$f = \alpha k_0 x + (1 - \alpha) k_0 z \quad (\text{A5-1})$$

$$\dot{z} = \frac{A\dot{x} - \left\{ \beta |\dot{x}| |z|^{N-1} z + \gamma \dot{x} |z|^N \right\} \nu}{\eta} \quad (\text{A5-2})$$

where,  $\beta, \gamma$ , and  $N$  are parameters that control the shape of the hysteresis loop, while  $A, \nu$ , and  $\eta$  are variables that control the material degradation.

From the yield deformation,  $\delta_y$ , the parameters  $\beta, \gamma$  are expressed as,

$$\beta = \beta_0 / \delta_y^N \quad \text{and} \quad \gamma = \gamma_0 / \delta_y^N \quad (\text{A5-3})$$

The model can be written as,

$$\dot{z} = \frac{A - |z|^N \{ \beta \operatorname{sgn}(\dot{x}z) + \gamma \} \nu}{\eta} \dot{x} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} \quad (\text{A5-4})$$

This leads to the following expression for the continuum tangent

$$k = \frac{\partial f}{\partial x} = \alpha k_0 + (1 - \alpha) k_0 \frac{\partial z}{\partial x} = \alpha k_0 + (1 - \alpha) k_0 \frac{A - |z|^N \{\beta \operatorname{sgn}(\dot{x}z) + \gamma\} \nu}{\eta} \quad (\text{A5-5})$$

The evolution of material degradation is governed by the following choice of equations (Baber and Noori 1985):

$$A = A_0 - \delta_A e, \quad \nu = 1 + \delta_\nu e, \quad \eta = 1 + \delta_\eta e \quad (\text{A5-6})$$

where  $e$  is defined by the rate equation

$$\dot{e} = (1 - \alpha) k_0 z \dot{x} \quad (\text{A5-7})$$

and  $A_0, \delta_A, \delta_\nu$ , and  $\delta_\eta$  are user-defined parameters.

#### b) Incremental form for numerical analysis

Incremental form of Eq.(A5-1) is

$$f_{(n+1)} = \alpha k_0 x_{(n+1)} + (1 - \alpha) k_0 z_{(n+1)} \quad (\text{A5-8})$$

By a backward Euler solution,

$$\begin{aligned} z_{(n+1)} &= z_{(n)} + \Delta t \dot{z}(t_{n+1}) \\ x_{(n+1)} &= x_{(n)} + \Delta t \dot{x}(t_{n+1}) \end{aligned} \quad (\text{A5-9})$$

Applied to Eq. (A5-4),

$$z_{(n+1)} = z_{(n)} + \Delta t \frac{A_{(n+1)} - |z_{(n+1)}|^N \left\{ \beta \operatorname{sgn} \left( \frac{x_{(n+1)} - x_{(n)}}{\Delta t} z_{(n+1)} \right) + \gamma \right\} \nu_{(n+1)}}{\eta_{(n+1)}} \frac{x_{(n+1)} - x_{(n)}}{\Delta t} \quad (\text{A5-10})$$

where

$$A_{(n+1)} = A_0 - \delta_A e_{(n+1)}, \quad \nu_{(n+1)} = 1 + \delta_\nu e_{(n+1)}, \quad \eta_{(n+1)} = 1 + \delta_\eta e_{(n+1)} \quad (\text{A5-11})$$

$$\begin{aligned} e_{(n+1)} &= e_{(n)} + \Delta t (1 - \alpha) k_0 z_{(n+1)} \frac{x_{(n+1)} - x_{(n)}}{\Delta t} \\ &= e_{(n)} + (1 - \alpha) k_0 z_{(n+1)} (x_{(n+1)} - x_{(n)}) \end{aligned} \quad (\text{A5-12})$$

Since

$$\operatorname{sgn} \left( \frac{x_{(n+1)} - x_{(n)}}{\Delta t} z_{(n+1)} \right) = \operatorname{sgn} \left( (x_{(n+1)} - x_{(n)}) z_{(n+1)} \right) \quad (\text{A5-13})$$

$$f(z_{(n+1)}) = z_{(n+1)} - z_{(n)} - \frac{\Phi}{\eta_{(n+1)}}(x_{(n+1)} - x_{(n)}) = 0 \quad (\text{A5-14})$$

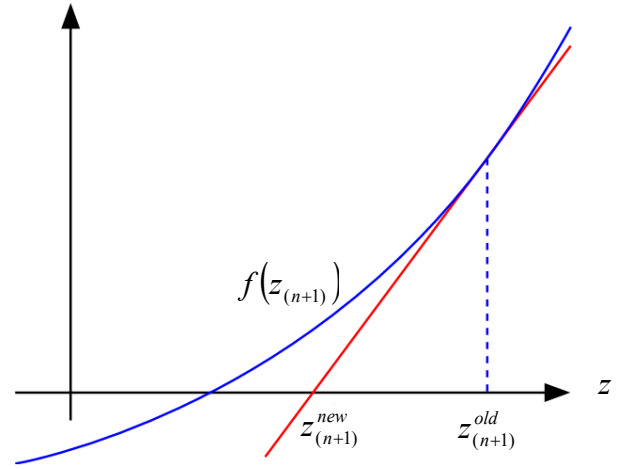
$$\Phi = A_{(n+1)} - |z_{(n+1)}|^N \Psi v_{(n+1)} \quad (\text{A5-15})$$

$$\Psi = \beta \operatorname{sgn}\{(x_{(n+1)} - x_{(n)})z_{(n+1)}\} + \gamma \quad (\text{A5-16})$$

A Newton-Raphson method is applied to solve the nonlinear equation  $f(z_{(n+1)}) = 0$ ,

$$z_{(n+1)}^{new} = z_{(n+1)}^{old} - \frac{f(z_{(n+1)})}{f'(z_{(n+1)})} \quad (\text{A5-17})$$

where the prime  $f'(z_{(n+1)})$  denotes derivative with respect to  $z_{(n+1)}$ ,



Evaluation of the function derivatives is summarized below.

Original $f(z_{(n+1)})$	Function derivatives $f'(z_{(n+1)})$
$e_{(n+1)} = e_{(n)} + \Delta t(1 - \alpha)k_0 \frac{x_{(n+1)} - x_{(n)}}{\Delta t}$	$e'_{(n+1)} = \Delta t(1 - \alpha)k_0 \frac{x_{(n+1)} - x_{(n)}}{\Delta t}$
$A_{(n+1)} = A_0 - \delta_A e_{(n+1)}$	$A'_{(n+1)} = -\delta_A e'_{(n+1)}$
$v_{(n+1)} = 1 + \delta_v e_{(n+1)}$	$v'_{(n+1)} = \delta_v e'_{(n+1)}$
$\eta_{(n+1)} = 1 + \delta_\eta e_{(n+1)}$	$\eta'_{(n+1)} = \delta_\eta e'_{(n+1)}$
$\Phi = A_{(n+1)} -  z_{(n+1)} ^N \Psi v_{(n+1)}$	$\Phi' = A'_{(n+1)} - N z_{(n+1)} ^{N-1} \operatorname{sgn}( z_{(n+1)} ) \Psi v_{(n+1)} -  z_{(n+1)} ^N \Psi v'_{(n+1)}$
$f(z_{(n+1)}) = z_{(n+1)} - z_{(n)} - \frac{\Phi}{\eta_{(n+1)}}(x_{(n+1)} - x_{(n)})$	$f'(z_{(n+1)}) = 1 - \frac{\Phi' \eta_{(n+1)} - \Phi \eta'_{(n+1)}}{\eta_{(n+1)}^2}(x_{(n+1)} - x_{(n)})$

(A5-18)

The procedure can now be summarized as follows:

1. While  $(|z_{(n+1)}^{new} - z_{(n+1)}^{old}| > tol)$

(a) Evaluate function

$$\begin{aligned}
 e_{(n+1)} &= e_{(n)} + (1 - \alpha) k_0 z_{(n+1)} (x_{(n+1)} - x_{(n)}) \\
 A_{(n+1)} &= A_0 - \delta_A e_{(n+1)}, \quad \nu_{(n+1)} = 1 + \delta_\nu e_{(n+1)}, \quad \eta_{(n+1)} = 1 + \delta_\eta e_{(n+1)} \\
 \Psi &= \beta \operatorname{sgn}\{(x_{(n+1)} - x_{(n)}) z_{(n+1)}\} + \gamma \\
 \Phi &= A_{(n+1)} - |z_{(n+1)}|^N \Psi \nu_{(n+1)} \\
 f(z_{(n+1)}) &= z_{(n+1)} - z_{(n)} - \frac{\Phi}{\eta_{(n+1)}} (x_{(n+1)} - x_{(n)})
 \end{aligned} \tag{A5-19}$$

(b) Evaluate function derivatives

$$\begin{aligned}
 e'_{(n+1)} &= \Delta t (1 - \alpha) k_0 \frac{x_{(n+1)} - x_{(n)}}{\Delta t} \\
 A'_{(n+1)} &= -\delta_A e'_{(n+1)} \\
 \nu'_{(n+1)} &= \delta_\nu e'_{(n+1)} \\
 \eta'_{(n+1)} &= \delta_\eta e'_{(n+1)} \\
 \Phi' &= A'_{(n+1)} - N |z_{(n+1)}|^{N-1} \operatorname{sgn}(z_{(n+1)}) \Psi \nu_{(n+1)} - |z_{(n+1)}|^N \Psi \nu'_{(n+1)} \\
 f'(z_{(n+1)}) &= 1 - \frac{\Phi' \eta_{(n+1)} - \Phi \eta'_{(n+1)}}{\eta_{(n+1)}^2} (x_{(n+1)} - x_{(n)})
 \end{aligned} \tag{A5-20}$$

(c) Obtain trial value in the Newton-Raphson scheme

$$z_{(n+1)}^{new} = z_{(n+1)} - \frac{f(z_{(n+1)})}{f'(z_{(n+1)})} \tag{A5-21}$$

(d) Update  $z_{(n+1)}$

$$z_{(n+1)}^{old} = z_{(n+1)} \quad \text{and} \quad z_{(n+1)} = z_{(n+1)}^{new} \tag{A5-22}$$

### c) Tangent stiffness

The tangent stiffness is necessary to compute the nonlinear structural analysis.

From the incremental forms:

$$f_{(n+1)} = \alpha k_0 x_{(n+1)} + (1 - \alpha) k_0 z_{(n+1)}$$

$$z_{(n+1)} = z_{(n)} + \Delta t \frac{A_{(n+1)} - |z_{(n+1)}|^N \left\{ \beta \operatorname{sgn} \left( \frac{x_{(n+1)} - x_{(n)}}{\Delta t} z_{(n+1)} \right) + \gamma \right\} v_{(n+1)}}{\eta_{(n+1)}} \frac{x_{(n+1)} - x_{(n)}}{\Delta t}$$

The tangent stiffness is calculated as (T. Haukaas and A. D. Kiureghian, 2004);

$$k_{(n+1)} = \frac{\partial f_{(n+1)}}{\partial x_{(n+1)}} = \alpha k_0 + (1 - \alpha) k_0 \frac{\partial z_{(n+1)}}{\partial x_{(n+1)}} \quad (\text{A5-23})$$

$$\frac{\partial z_{(n+1)}}{\partial x_{(n+1)}} = \frac{b_4}{b_5} \quad (\text{A5-24})$$

where

$$\Psi = \beta \operatorname{sgn}((x_{(n+1)} - x_{(n)}) z_{(n+1)}) + \gamma$$

$$\Phi = A_{(n+1)} - |z_{(n+1)}|^N \Psi v_{(n+1)}$$

$$b_1 = (1 - \alpha) k_0 z_{(n+1)}$$

$$b_2 = (1 - \alpha) k_0 (x_{(n+1)} - x_{(n)})$$

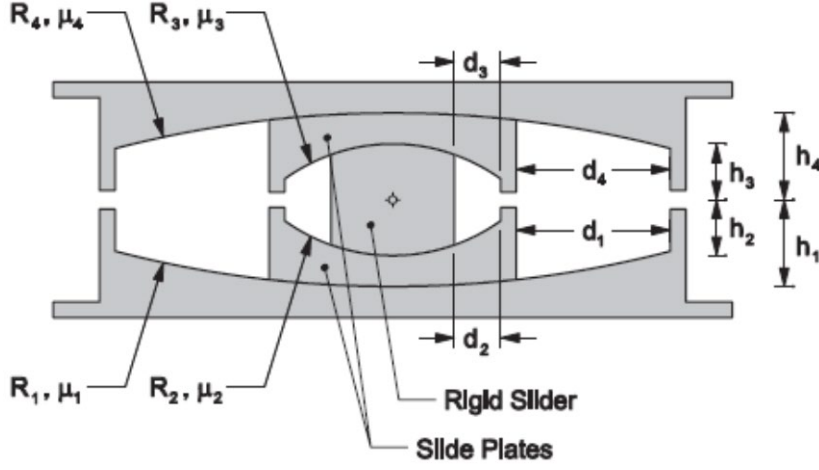
$$b_3 = \frac{(x_{(n+1)} - x_{(n)})}{\eta_{(n+1)}}$$

$$b_4 = -b_3 \delta_A b_1 - b_3 |z_{(n+1)}|^N \Psi \delta_v b_1 - \frac{\Phi}{\eta_{(n+1)}^2} (x_{(n+1)} - x_{(n)}) \delta_\eta b_1 + \frac{\Phi}{\eta_{(n+1)}}$$

$$b_5 = 1 + b_3 \delta_A b_2 + b_3 N |z_{(n+1)}|^{N-1} \operatorname{sgn}(z_{(n+1)}) \Psi v_{(n+1)} \\ + b_3 |z_{(n+1)}|^N \Psi \delta_v b_2 + \frac{\Phi}{\eta_{(n+1)}^2} (x_{(n+1)} - x_{(n)}) \delta_\eta b_2$$

## A-6. Hysteresis of FPB (Friction Pendulum Bearing)

### 1) TPB (Triple Friction Pendulum Bearing)



#### Reference

- 1) Daniel M. Fenz, Michael C. Constantinou, "Spherical sliding isolation bearings with adaptive behavior: Theory", Earthquake Engineering and Structural Dynamics, 2—8: 37: 163-183

#### Effective radius of each surface

$$R_{eff1} = R_1 - h_1, \quad R_{eff2} = R_2 - h_2, \quad R_{eff3} = R_3 - h_3, \quad R_{eff4} = R_4 - h_4$$

#### Friction force of each surface

$$F_{f1} = \mu_1 W, \quad F_{f2} = \mu_2 W, \quad F_{f3} = \mu_3 W, \quad F_{f4} = \mu_4 W$$

#### Stiffness after sliding of each surface

$$K_{f1} = \frac{W}{R_{eff1}}, \quad K_{f2} = \frac{W}{R_{eff2}}, \quad K_{f3} = \frac{W}{R_{eff3}}, \quad K_{f4} = \frac{W}{R_{eff4}}$$

where,

$R_i$ : the radius of curvature of the  $i$ -th sliding surface,

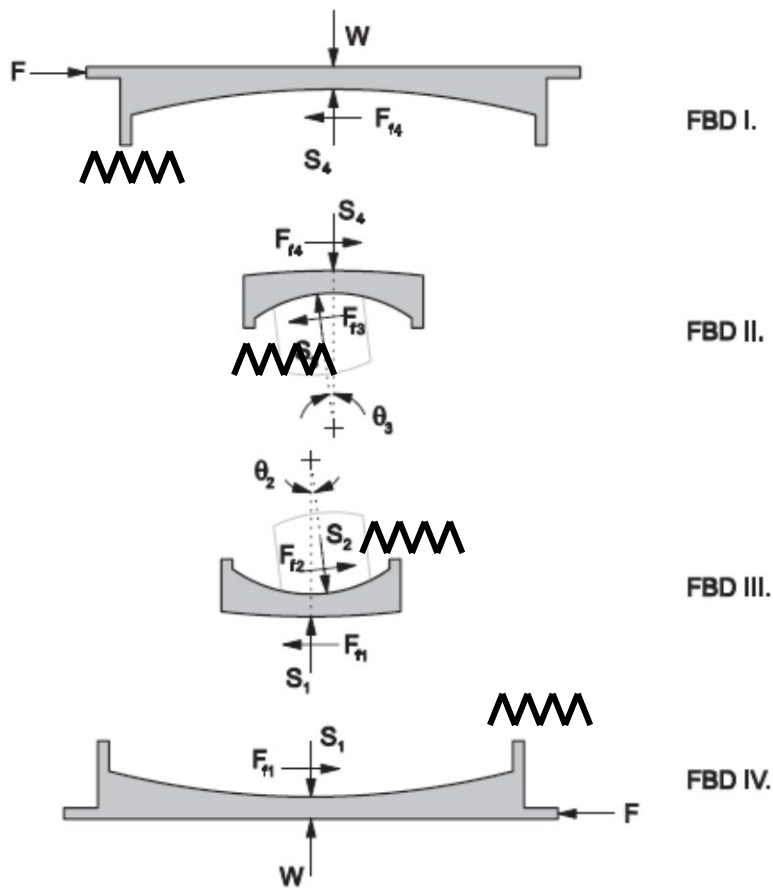
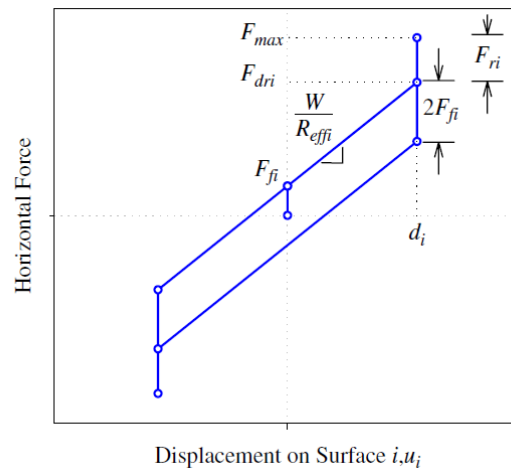
$h_i$ : the radial distance between the  $i$ -th sliding surface and the pivot point of the articulated slider

$d_i$ : the displacement capacity to the displacement restrainer on the  $i$ -th sliding surface,

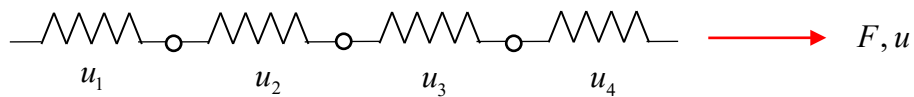
$\mu_i$ : the coefficient of friction of the  $i$ -th sliding surface,

$W$ : the vertical load.

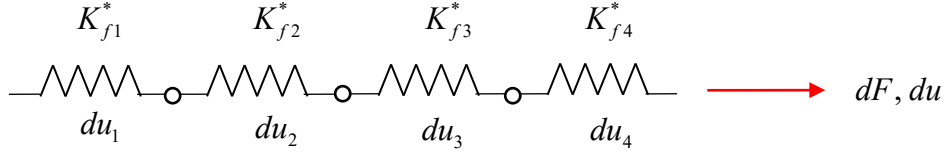
Restoring force of each surface is the bilinear model with the capacity deformation.



The TPB can be modeled as a model of four springs with bilinear restoring forces connected in series.



In small incremental deformation  $du_i$  ( $i = 1, \dots, 4$ ), if the instantaneous stiffness of each spring  $K_{fi}^*$  is used,



The total incremental deformation  $du$  of the TPB is the sum of the deformation of each spring

$$du = \sum_{i=1}^4 du_i = du_1 + du_2 + du_3 + du_4 = dF \left( \frac{1}{K_{f1}^*} + \frac{1}{K_{f2}^*} + \frac{1}{K_{f3}^*} + \frac{1}{K_{f4}^*} \right)$$

Therefore, the deformation of each spring  $u_i$  is obtained from the total deformation  $u$  as

$$dF = K^* du, \quad K^* = \frac{1}{\left( \frac{1}{K_{f1}^*} + \frac{1}{K_{f2}^*} + \frac{1}{K_{f3}^*} + \frac{1}{K_{f4}^*} \right)}$$

The conditions of parameters are:

- 1) Effective radii of inner surfaces 2 and 3 are smaller than those of outer surfaces 1 and 4

$$R_{eff1} = R_{eff4} \gg R_{eff2} = R_{eff3}$$

- 2) Inner surfaces 2 and 3 slide before outer surfaces 1 and 4

$$\mu_2 = \mu_3 < \mu_1 < \mu_4$$

- 3) For the outer surface 1 to slip before the inner capacity deformation  $d_2$  is reached, i.e., the force on face 1 is less than the force on face 2 at the capacity deformation, then

$$F_{f1} < \frac{W}{R_{eff2}} d_2 + F_{f2} \rightarrow \mu_1 W < \frac{W}{R_{eff2}} d_2 + \mu_2 W \rightarrow d_2 > (\mu_1 - \mu_2) R_{eff2}$$

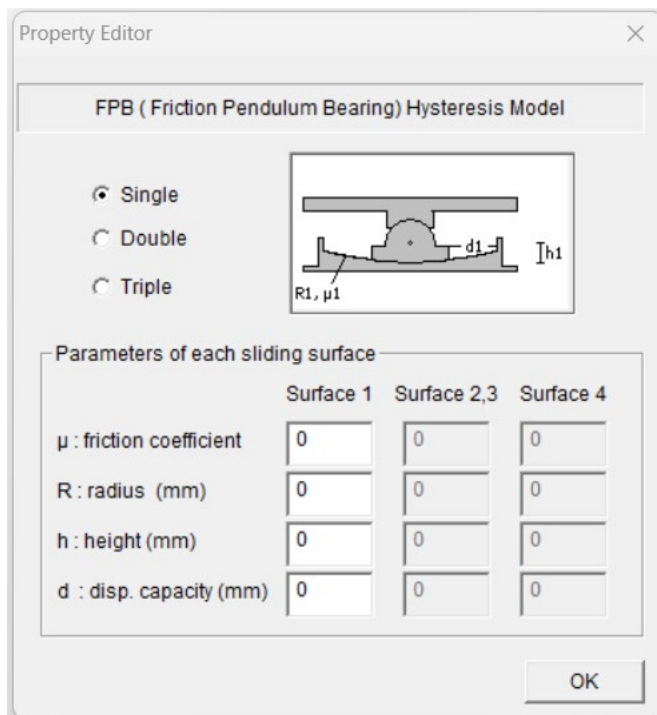
- 4) For the outer surface 4 to slip before the inner capacity deformation  $d_3$  is reached, i.e., the force on face 4 is less than the force on face 3 at the capacity deformation, then

$$F_{f4} < \frac{W}{R_{eff3}} d_3 + F_{f3} \rightarrow d_3 > (\mu_4 - \mu_3) R_{eff3}$$

- 5) For face 4, which has the greatest frictional force, to slip before the capacity deformation of face 1 is reached

$$F_{f4} < \frac{W}{R_{eff1}} d_1 + F_{f1} \rightarrow d_1 > (\mu_4 - \mu_1) R_{eff1}$$

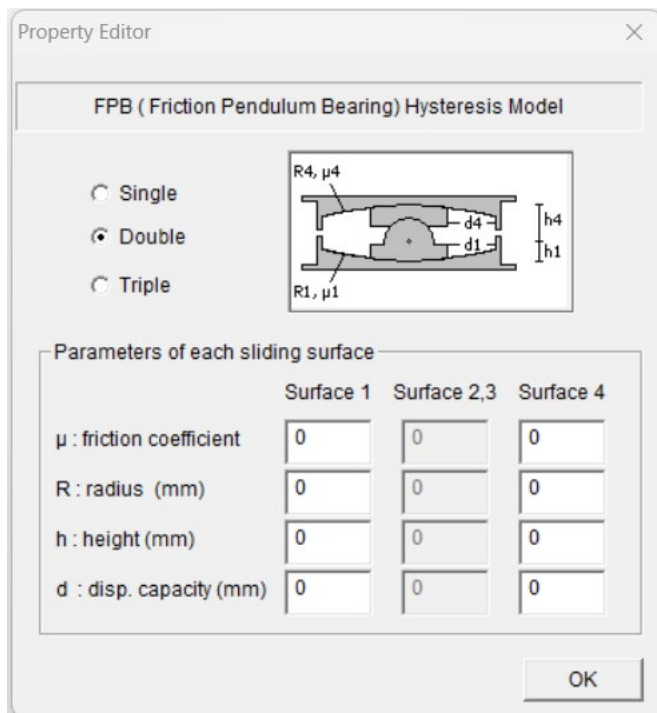
## 2) Single Friction Pendulum Bearing



In case of a single sliding surface, the force-deformation of FPB is

$$dF = K^* du, \quad K^* = K_{f1}^*$$

## 3) Double Friction Pendulum Bearing



In case of double sliding surfaces, the force-deformation of FPB is

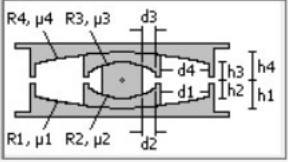
$$dF = K^* du, \quad K^* = \frac{1}{\left( \frac{1}{K_{f1}^*} + \frac{1}{K_{f4}^*} \right)}$$

Example)

Property Editor

FPB (Friction Pendulum Bearing) Hysteresis Model

☐ Single  
☐ Double  
☒ Triple



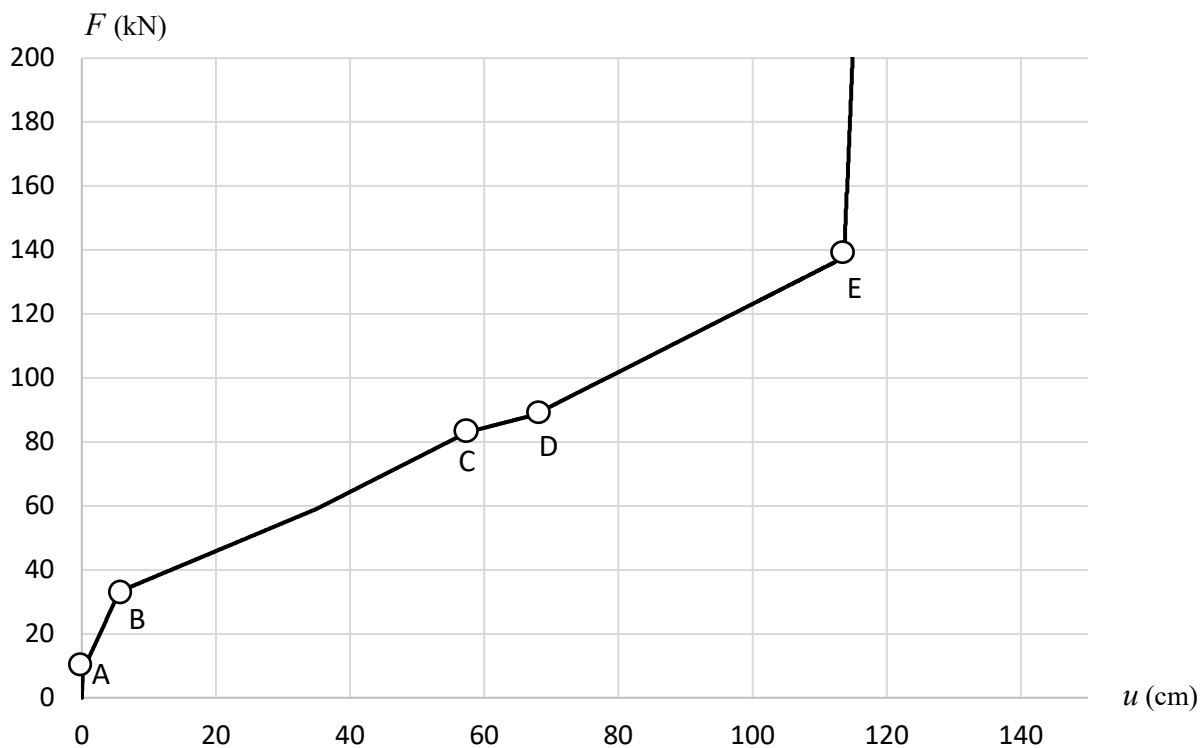
Parameters of each sliding surface

	Surface 1	Surface 2,3	Surface 4
$\mu$ : friction coefficient	0.08	0.02	0.08
$R$ : radius (mm)	3962	457	3962
$h$ : height (mm)	114	38	114
$d$ : disp. capacity (mm)	514	51	514

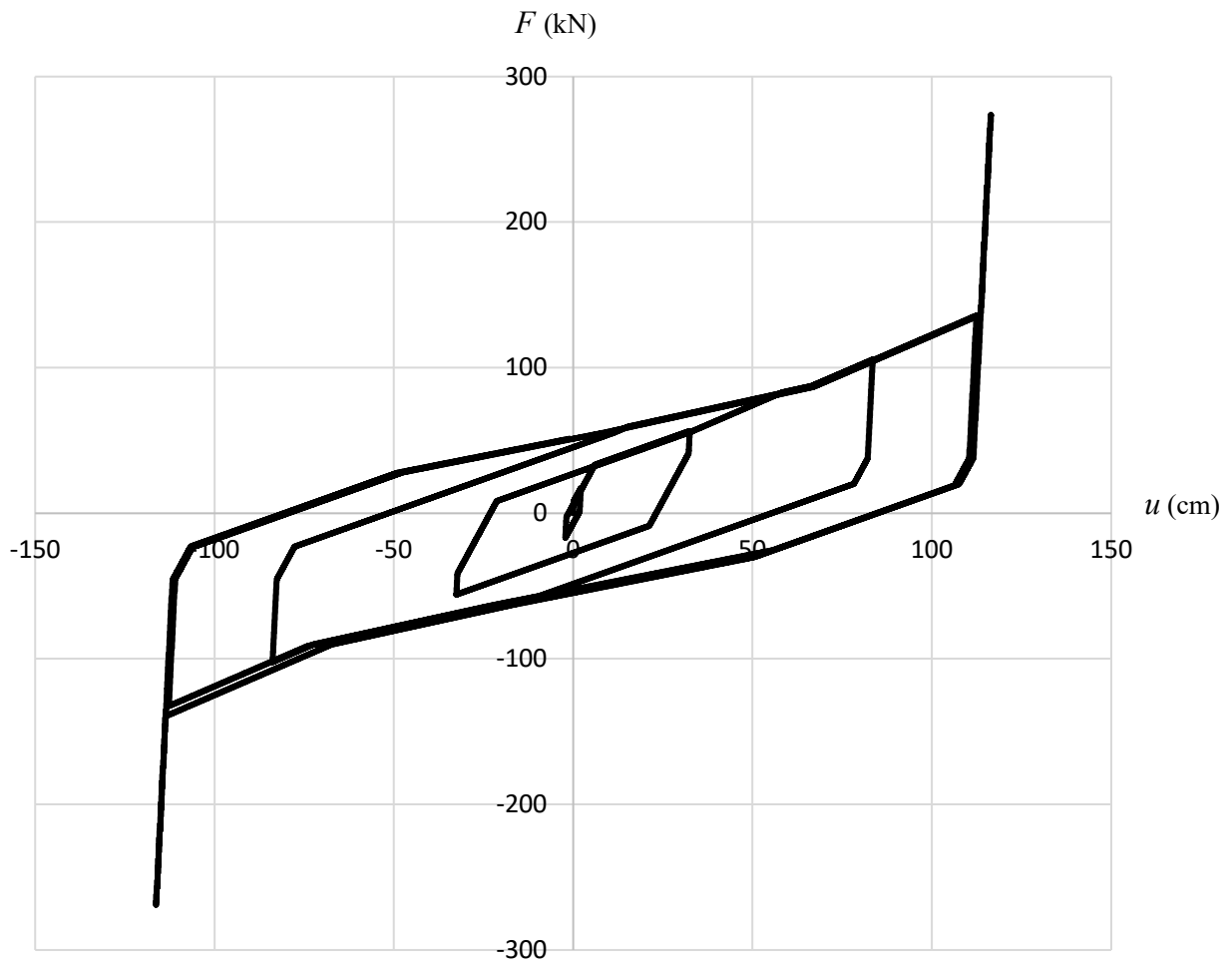
OK

$W = 415.12 \text{ kN}$

Note) In STERA, the frictional bearing capacity is calculated from the initial vertical load and the coefficient of friction. Note that the frictional capacity remains the same even if the axial force varies.



	$F$	$u$
A	$F_{f2} = F_{f3}$	0
B	$F_{f1}$	$u^* = (\mu_1 - \mu_2)R_{eff2} + (\mu_1 - \mu_3)R_{eff3}$
C	$F_{f4}$	$u^{**} = \mu_4(R_{eff1} + R_{eff3}) + (\mu_1 - \mu_2)R_{eff2} - \mu_1R_{eff1} - \mu_3R_{eff3}$
D	$F_{dr1} = \frac{W}{R_{eff1}}d_1 + F_{f1}$	$u_{dr1} = u^{**} + d_1 \left( 1 + \frac{R_{eff4}}{R_{eff1}} \right) - (\mu_4 - \mu_1)(R_{eff1} + R_{eff4})$
E	$F_{dr4} = \frac{W}{R_{eff4}}d_4 + F_{f4}$	$u_{dr4} = u_{dr1} + \left[ \left( \frac{d_4}{R_{eff4}} + \mu_4 \right) - \left( \frac{d_1}{R_{eff1}} + \mu_1 \right) \right] (R_{eff2} + R_{eff4})$



### 3.6 Masonry Wall

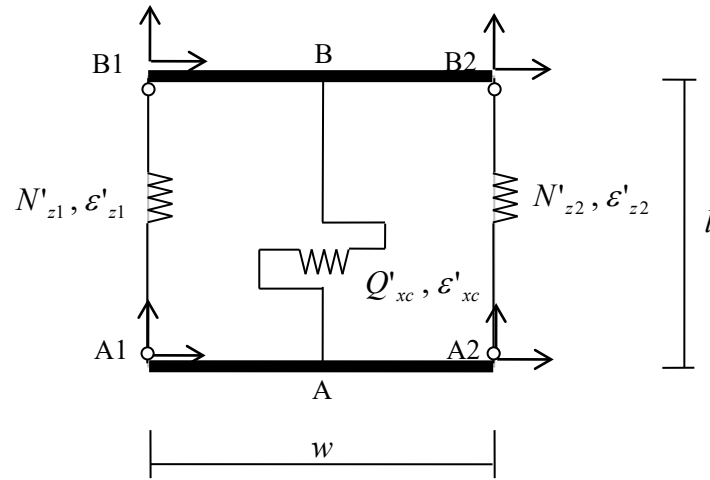


Figure 3-6-1 Element model for masonry wall

#### a) Nonlinear shear spring

Hysteresis model of the nonlinear shear spring is defined as the poly-linear slip model as shown in Figure 3-6-2.

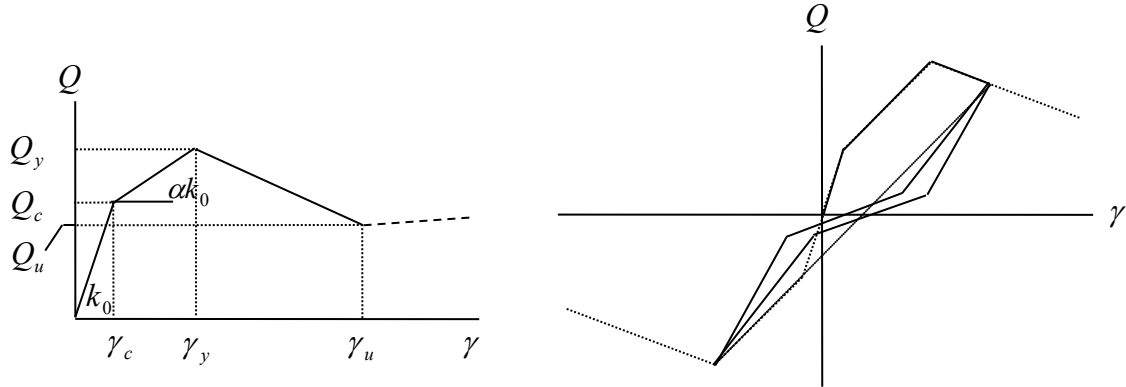
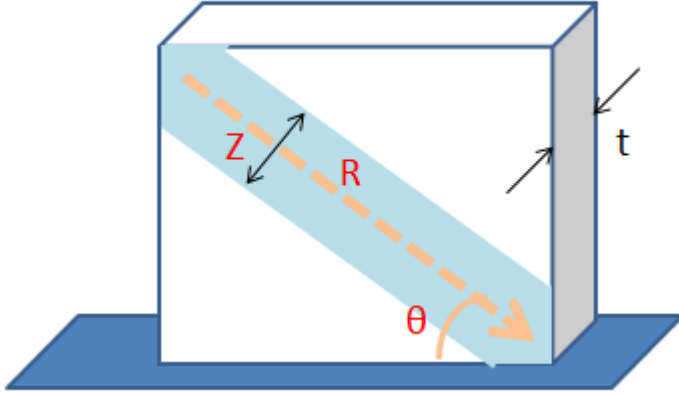


Figure 3-6-2 Hysteresis model of the nonlinear shear spring

The characteristic values,  $Q_c, Q_y, Q_u$  are obtained based on the formulation described in the reference (Paulay and Priestley, 1992).

The procedure to obtain the shear strength is shown below:

### (1) Compression strength of masonry prism



Compression strength of diagonal strut is

$$R = Z t f'_m \quad (3-6-1)$$

where,

$f'_m$	:	Compression strength of the masonry prism
$Z$	:	Width of the diagonal strut ( $Z = 0.25 d$ , $d$ is diagonal length)
$t$	:	Thickness of wall

The compression strength of the masonry prism ( $f'_m$ ) is determined by the following equation (Paulay and Priestley, 1992),

$$f'_m = \frac{f'_{cb} (f'_{tb} + \alpha f'_j)}{U_u (f'_{tb} + \alpha f'_{cb})} \quad (3-6-2)$$

$$\alpha = \frac{j}{4.1 h_b} \quad (3-6-3)$$

where,

$f'_{cb}$	:	Compressive strength of the brick
$f'_{tb}$	:	Tensile strength of the brick ( $= 0.1 f'_{cb}$ )
$f'_j$	:	Compressive strength of the mortar
$j$	:	Mortar joint thickness
$h_b$	:	Height of masonry unit
$U_u$	:	Stress non-uniformity coefficient ( $= 1.5$ )

Another formula is proposed by Eurocode 6:

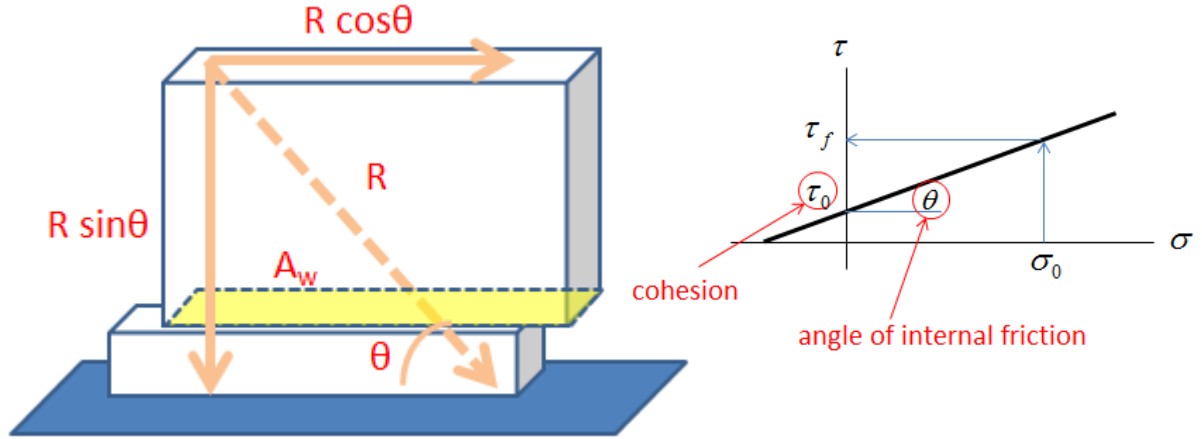
$$f'_m = k \cdot (f'_{cb})^a \cdot (f'_j)^b \quad (3-6-4)$$

where,  $k, \alpha, \beta$  : constants provided by the table in Eurocode 6

The shear strength is then obtained as,

$$V_c = R \cos \theta = Z t f'_m \cos \theta \quad (3-6-5)$$

## (2) Shear strength by sliding shear failure



The maximum shear stress is obtained from the Mohr-Coulomb criterion:

$$\tau_f = \tau_0 + \mu \sigma_0 = \tau_0 + \tan \phi \sigma_0 \quad (3-6-6)$$

where,

$\tau_0$  : Cohesive capacity of the mortar beds ( $=0.04 f'_m$ ) (Paulay and Priestly, 1992)

$\mu$  : Sliding friction coefficient along the bed joint

$$\mu = 0.654 + 0.000515 f'_j \quad (\text{Chen et.al, 2003, } f'_j (\text{kg} / \text{cm}^2))$$

$\sigma_0$  : Compression stress ( $= W / A_w = R \sin \theta / A_w$ )

The shear strength is

$$V_f = \tau_f A_w = \left( \tau_0 + \mu \frac{W}{A_w} \right) A_w = \tau_0 A_w + \mu W \quad (3-6-7)$$



Substituting  $V_f = R \cos \theta$ ,  $W = R \sin \theta$

where  $\theta$  is an angle subtended by diagonal strut to horizontal plane

$$R \cos \theta = \tau_0 A_w + \mu R \sin \theta$$

$$R \cos \theta (1 - \mu \tan \theta) = \tau_0 A_w \quad (3-6-8)$$

$$R \cos \theta = \frac{\tau_0 A_w}{1 - \mu \tan \theta}$$

Therefore,

$$V_f = \frac{\tau_0 A_w}{1 - \mu \tan \theta} \quad (3-6-9)$$

### (3) Characteristic values of nonlinear skeleton

The shear resistance,  $Q_y$ , is calculated to be the minimum value between the shear strength by sliding shear failure,  $V_f$ , and the shear strength of diagonal compression failure,  $V_c$ , that is,

$$Q_y = \min(V_f, V_c) \quad (3-6-10)$$

The shear displacement at the maximum resistance,  $\gamma_y$ , is obtained as (Madan et al., 1997),

$$\gamma_y = \frac{\varepsilon'_m d_m}{\cos \theta} \quad (3-6-11)$$

where,

$$\varepsilon'_m : \text{Compression strain at the maximum compression stress} \\ (\varepsilon'_m = 0.0018, \text{ Hossein and Kabeyasawa, 2004})$$

Initial elastic stiffness is assumed as (Madan et al., 1997)

$$k_0 = 2Q_y / \gamma_y \quad (3-6-12)$$

From Figure 3-6-2, the shear resistance at crack,  $Q_c$ , is obtained as,

$$Q_c = \frac{Q_y - \alpha k_0 \gamma_y}{1 - \alpha} \quad (3-6-13)$$

where,  $\alpha$  is the stiffness ratio of the second stiffness and assumed to be 0.2.

Shear displacement at crack is then obtained as,

$$\gamma_c = Q_c / k_0 \quad (3-6-14)$$

Shear resistance and displacement at the ultimate stage are assumed as (Hossein & Kabeyasawa, 2004)

$$Q_u = 0.3Q_y \quad (3-6-15)$$

$$\gamma_u = 3.5(0.01h_m - \gamma_y) \quad (3-6-16)$$

where,  $h_m$  is the height of masonry wall.

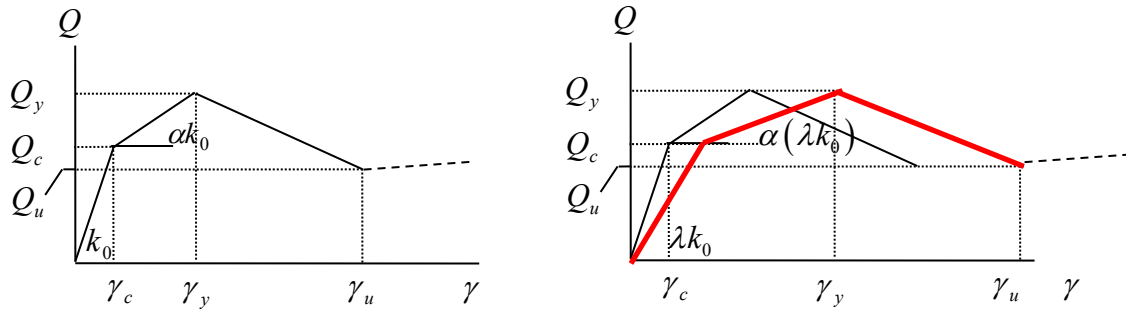
### References:

- 1) T. Pauley, M.J.N. Priestley, 1992, Seismic Design of Reinforced Concrete and Masonry building, JOHN WILEY & SONS, INC.
- 2) Hossein Mostafaei, Toshimi Kabeyasawa, 2004, Effect of Infill Walls on the Seismic Response of Reinforced Concrete Buildings Subjected to the 2003 Bam Earthquake Strong Motion : A Case Study of Bam Telephone Centre, Bulletin Earthquake Research Institute, The university of Tokyo
- 3) A. Madan, A.M. Reinhorn, J. B. Mandar, R.E. Valles, 1997, Modeling of Masonry Infill Panels for Structural Analysis, Journal of Structural Division, ASCE, Vol.114, No.8, pp.1827-1849

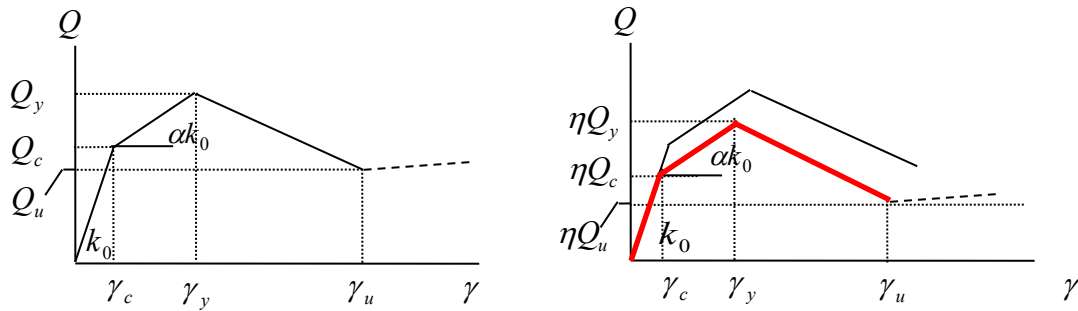
#### (4) Modification factors

STERA\_3D provides modification factors for stiffness and shear strength in the option menu.

The stiffness modification factor,  $\lambda$ , changes the stiffness while maintaining the shear strength in the skeletal curve.



The strength modification factor,  $\eta$ , changes the strength while maintaining the stiffness in the skeletal curve.



#### b) Vertical springs

For the moment, the vertical springs of the element model in Figure 3-6-1 are assumed to be elastic springs.

$$N'_{z1} = k_z \varepsilon'_{z1}, \quad N'_{z2} = k_z \varepsilon'_{z2} \quad (3-6-17)$$

$$k_z = E_m (tl_w) / 2 \quad (3-6-18)$$

where,

- $E_m$  : Modulus of elasticity of masonry prism ( $=550 f'_m$ , FEMA 356, 2000)
- $t$  : Thickness of masonry wall
- $l_w$  : Width of masonry wall

### 3.7 Passive Damper

#### a) Hysteresis damper

Hysteresis damper is modeled as a shear spring as shown in Figure 3-7-1.

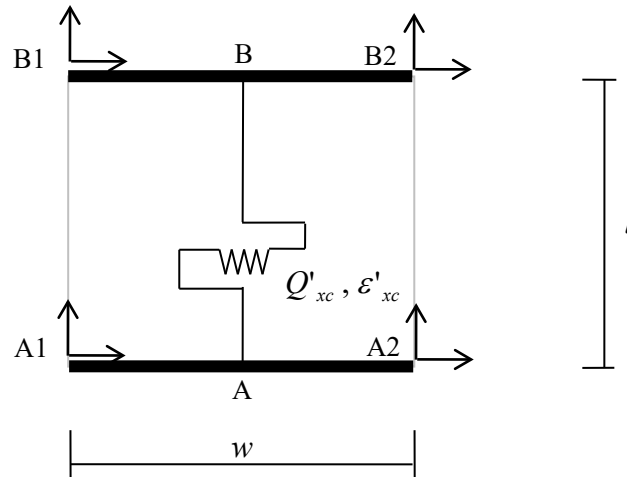
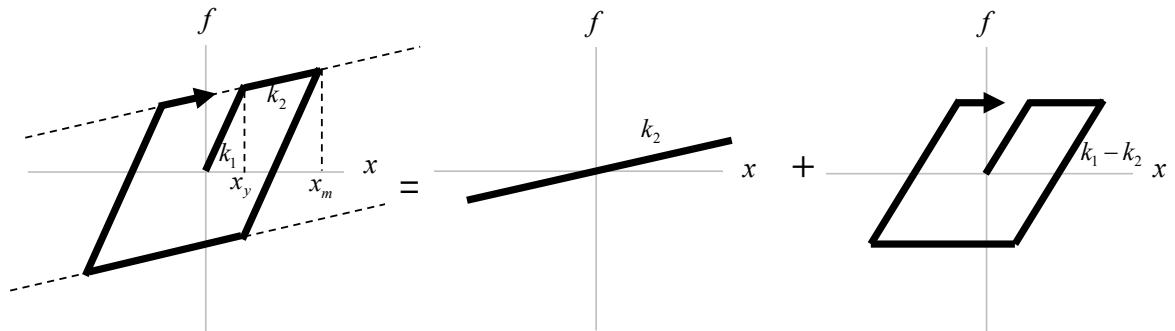


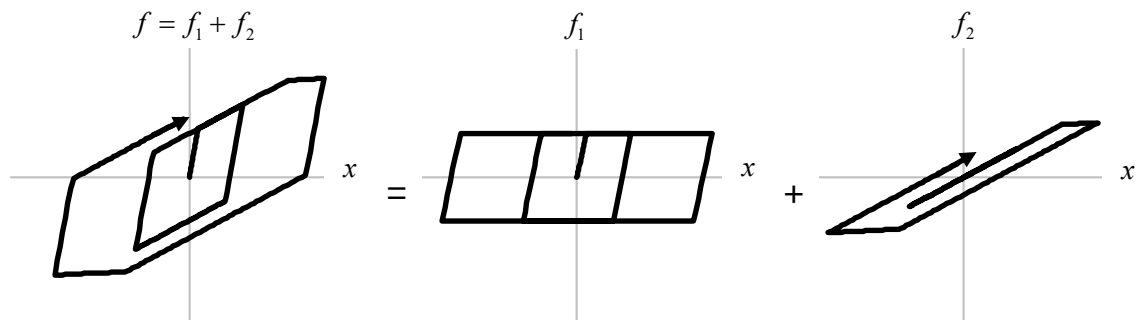
Figure 3-7-1 Element model for passive damper

Different types of hysteresis model are prepared for the force-deformation relationship of the spring.

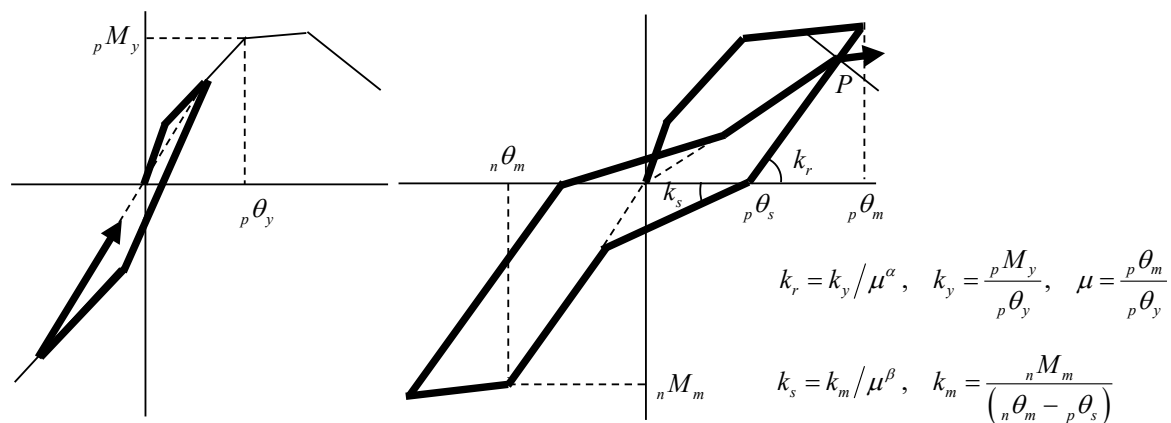
#### (1) Bi-linear Model



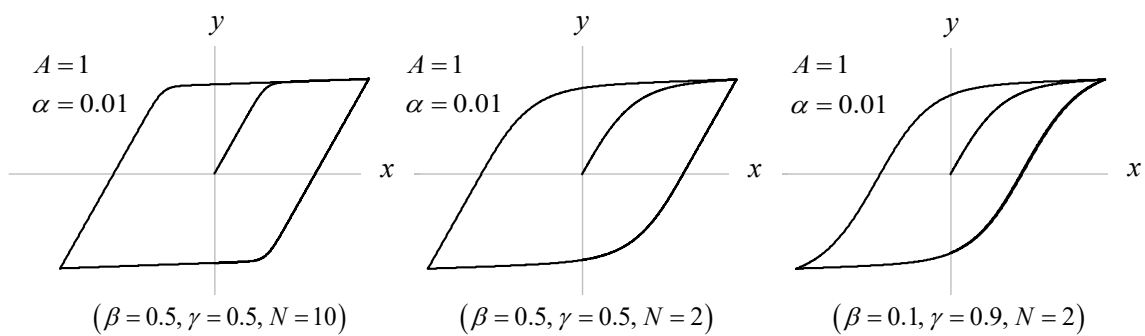
#### (2) Normal-trilinear Model



(3) Degrading Tri-linear model



(4) Bouc-Wen model



(5) Nonlinear Spring model

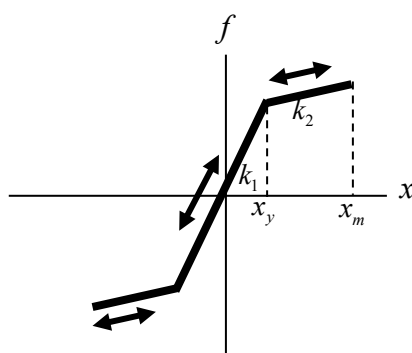


Figure 3-7-2 Hysteresis model of the shear spring

### b) Viscous damper

Viscous damper is modeled as a shear spring as shown in Figure 3-7-3.

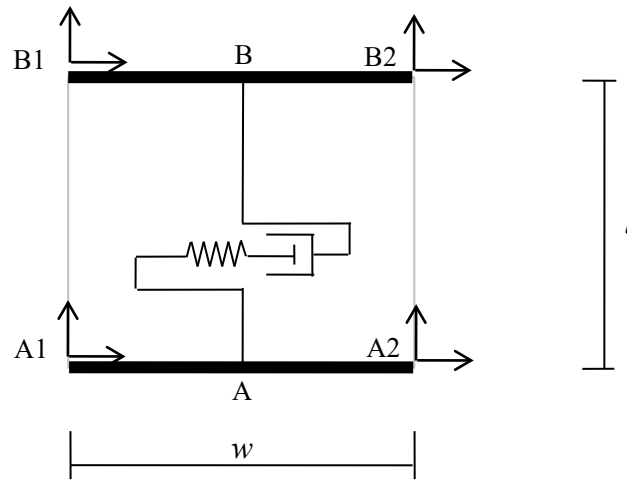


Figure 3-7-3 Element model for passive damper

#### (1) Algorithm for oil damper devise

Figure 3-7-4 shows the Maxwell model with an elastic spring with stiffness,  $K_d$ , and a dashpot with damping coefficient,  $C$ .

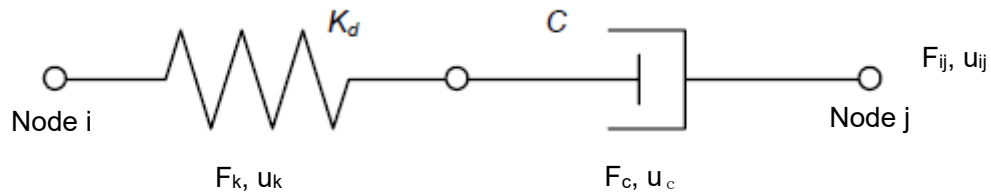


Figure 3-7-4 Maxwell model

Since the elastic spring and the dashpot are connected in a series,

$$F_k = F_c = F_{ij} \quad (3-7-1)$$

where,  $F_k$  : force of the elastic spring

$F_c$  : force of the dashpot

$F_{ij}$  : force between i-j nodes

The force of the elastic spring,  $F_k$ , is obtained as,

$$F_k = K_d u_k = K_d (u_{ij} - u_c) \quad (3-7-2)$$

where,  $u_k$  : relative displacement of the elastic spring  
 $u_c$  : relative displacement of the dashpot  
 $u_{ij}$  : relative displacement between i-j nodes

For an oil damper, the force-velocity relationship of the dashpot is defined as shown in Figure 3-7-5.

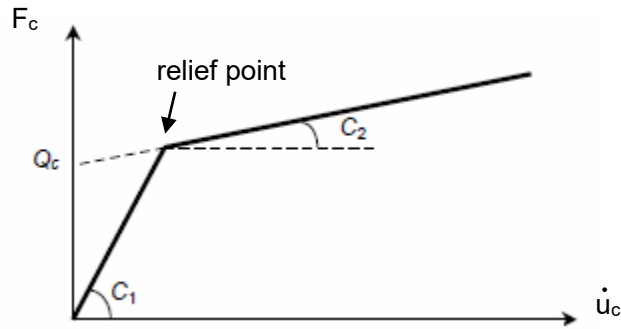


Figure 3-7-5 Dashpot element

The force of the dashpot after the relief point is,

$$F_c = C_2 \dot{u}_c + Q_c \quad (3-7-3)$$

Substituting Equations (3-7-2) and (3-7-3) into (3-7-1)

$$K_d (u_{ij} - u_c) = C_2 \dot{u}_c + Q_c \quad (3-7-4)$$

When the time interval  $\Delta t$  is small enough, the velocity at time  $t$  can be expressed as,

$$\dot{u}_c(t) = \frac{\Delta u_c(t)}{\Delta t} \quad (3-7-5)$$

$$\Delta u_c(t) = u_c(t) - u_c(t - \Delta t) \quad (3-7-6)$$

Substituting above equations into Equation (3-7-4),

$$\Delta u_c(t) = \frac{K_d (u_{ij}(t) - u_c(t - \Delta t)) - Q_c}{\frac{C_2}{\Delta t} + K_d} \quad (3-7-7)$$

The algorithm to obtain the force  $F_{ij}(t)$  from  $u_{ij}(t)$  is as follows:

- 1) Evaluate  $\Delta u_c(t)$  from Equation (3-7-7)
- 2) Evaluate  $u_c(t)$  from Equation (3-7-6)
- 3) Evaluate  $F_{ij}(t)$  from Equation (3-7-2)

Before the relief point of the dashpot, Equation (3-7-7) will be obtained by changing  $C_2 \rightarrow C_1$ ,  $Q_c = 0$  as

$$\Delta u_c(t) = \frac{K_d (u_{ij}(t) - u_c(t - \Delta t))}{\frac{C_1}{\Delta t} + K_d} \quad (3-7-8)$$

When the velocity of the dashpot is over the negative relief point, Equation (3-7-7) will be obtained by changing  $Q_c \rightarrow -Q_c$ ,

$$\Delta u_c(t) = \frac{K_d (u_{ij}(t) - u_c(t - \Delta t)) + Q_c}{\frac{C_2}{\Delta t} + K_d} \quad (3-7-9)$$

In case there is no elastic spring,

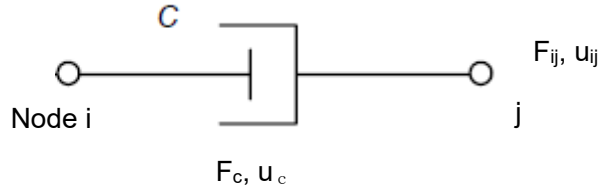


Figure 3-7-6 Dashpot element without elastic spring

$$u_{ij}(t) = u_c(t)$$

$$F_{uj} = F_c = C_2 \dot{u}_c + Q_c$$

$$\dot{u}_c(t) = \frac{\Delta u_c(t)}{\Delta t} = \frac{\Delta u_{ij}(t)}{\Delta t}$$

Therefore,

$$F_{ij}(t) = C_2 \frac{\Delta u_{ij}(t)}{\Delta t} + Q_c \quad (3-7-10)$$

Before the relief point of the dashpot,

$$F_{ij}(t) = C_1 \frac{\Delta u_{ij}(t)}{\Delta t} \quad (3-7-11)$$

When the velocity of the dashpot is over the negative relief point,

$$F_{ij}(t) = C_2 \frac{\Delta u_{ij}(t)}{\Delta t} - Q_c \quad (3-7-12)$$

(2) Algorithm for viscous damper devise

Figure 3-7-7 shows the Maxwell model with an elastic spring with stiffness,  $K_d$ , and a dashpot with damping coefficient,  $C$ .

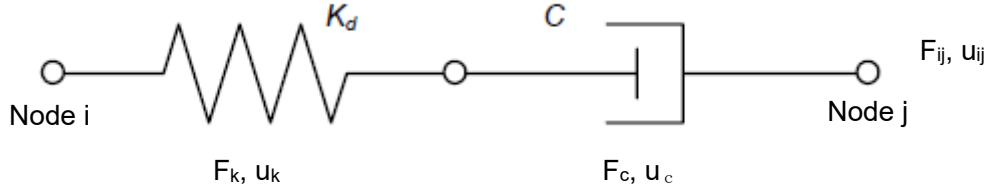


Figure 3-7-7 Maxwell model

Since the elastic spring and the dashpot are connected in a series,

$$F_k = F_c = F_{ij} \quad (3-7-13)$$

where,

- $F_k$  : force of the elastic spring
- $F_c$  : force of the dashpot
- $F_{ij}$  : force between i-j nodes

The force of the elastic spring,  $F_k$ , is obtained as,

$$F_k = K_d u_k = K_d (u_{ij} - u_c) \quad (3-7-14)$$

where,

- $u_k$  : relative displacement of the elastic spring
- $u_c$  : relative displacement of the dashpot
- $u_{ij}$  : relative displacement between i-j nodes

For a viscous damper, the force-velocity relationship of the dashpot is defined as shown in Figure 3-7-8,

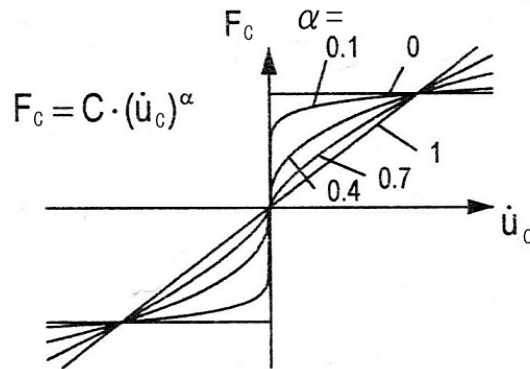


Figure 3-7-8 Dashpot element

That is,

$$F_c = C \operatorname{sgn}(\dot{u}_c(t)) |\dot{u}_c(t)|^\alpha \quad (3-7-15)$$

From Equations (3-7-13) and (3-7-14)

$$\frac{\dot{F}_{ij}(t)}{K_d} + u_c(t) = u_{ij}(t) \quad (3-7-16)$$

Taking time differential and substituting Equation (3-7-15) give

$$\frac{\dot{F}_{ij}(t)}{K_d} + \text{sgn}(F_{ij}(t)) \left( \frac{|F_{ij}(t)|}{C} \right)^{1/\alpha} = \dot{u}_{ij}(t) \quad (3-7-17)$$

The numerical integration method, Runge-Kutta Method, can be used to solve the Equation (3-7-17).

In general, the solution of the differential equation,  $\dot{y}(t) = f(y, t)$ , is obtained by Runge-Kutta Method as follows:

$$y_{n+1} = y_n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3) \quad (3-7-18)$$

$$k_0 = f(y_n, t_n)\Delta t$$

$$k_1 = f(y_n + k_0/2, t_n + \Delta t/2)\Delta t$$

$$k_2 = f(y_n + k_1/2, t_n + \Delta t/2)\Delta t$$

$$k_3 = f(y_n + k_2, t_n + \Delta t)\Delta t$$

Equation (3-7-17) can be written as

$$\dot{F}_{ij}(t) = \left( \dot{u}_{ij}(t) - \text{sgn}(F_{ij}(t)) \left( \frac{|F_{ij}(t)|}{C} \right)^{1/\alpha} \right) K_d \quad (3-7-19)$$

Applying Runge-Kutta Method gives the following algorithm,

$$F_{ij}(t_{n+1}) = F_{ij}(t_n) + \frac{1}{6}(k_0(t_n) + 2k_1(t_n) + 2k_2(t_n) + k_3(t_n)) \quad (3-7-20)$$

$$k_0 = \left( \dot{u}_{ij}(t_n) - \text{sgn}(F_{ij}(t_n)) \left( \frac{|F_{ij}(t_n)|}{C} \right)^{1/\alpha} \right) K_d \Delta t$$

$$k_1 = \left( \dot{u}_{ij}(t_n + \Delta t/2) - \text{sgn}(F_{ij}(t_n) + k_0/2) \left( \frac{|F_{ij}(t_n) + k_0/2|}{C} \right)^{1/\alpha} \right) K_d \Delta t$$

$$k_2 = \left( \dot{u}_{ij}(t_n + \Delta t/2) - \text{sgn}(F_{ij}(t_n) + k_1/2) \left( \frac{|F_{ij}(t_n) + k_1/2|}{C} \right)^{1/\alpha} \right) K_d \Delta t$$

$$k_3 = \left( \dot{u}_{ij}(t_n + \Delta t) - \text{sgn}(F_{ij}(t_n) + k_2) \left( \frac{|F_{ij}(t_n) + k_2|}{C} \right)^{1/\alpha} \right) K_d \Delta t$$

In this algorithm, it is assumed as,

$$\dot{u}_{ij}(t_n + \Delta t / 2) = \frac{\dot{u}_{ij}(t_n) + \dot{u}_{ij}(t_n + \Delta t)}{2} \quad (3-7-21)$$

In case there is no elastic spring,

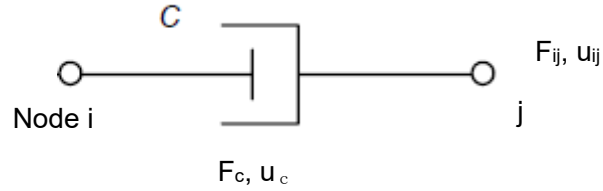


Figure 3-7-9 Dashpot element without elastic spring

$$u_{ij}(t) = u_c(t) \quad (3-7-22)$$

$$F_{ij} = F_c = C \operatorname{sgn}(\dot{u}_c(t)) |\dot{u}_c(t)|^\alpha \quad (3-7-23)$$

$$\dot{u}_c(t) = \frac{\Delta u_c(t)}{\Delta t} = \frac{\Delta u_{ij}(t)}{\Delta t} \quad (3-7-24)$$

Therefore,

$$F_{ij}(t) = C \operatorname{sgn}\left(\frac{\Delta u_{ij}(t)}{\Delta t}\right) \left|\frac{\Delta u_{ij}(t)}{\Delta t}\right|^\alpha \quad (3-7-25)$$

(3) Algorithm for visco-elastic damper device

Figure 3-7-10 shows the Voigt (or Kelvin-Voigt) model with an elastic spring with stiffness,  $K$ , and a dashpot with damping coefficient,  $C$ . The stiffness of the connection is represented as  $K_d$ .

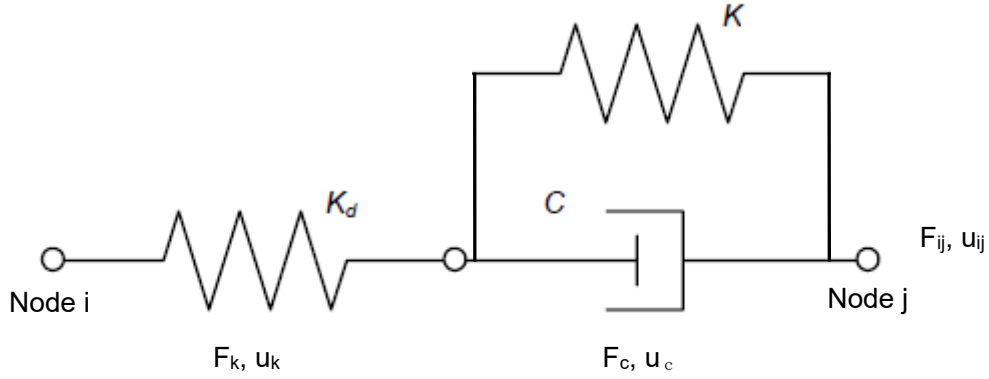


Figure 3-7-10 Voigt (or Kelvin-Voigt) model

Since the elastic spring and the dashpot are connected in a series,

$$F_k = F_c = F_{ij} \quad (3-7-26)$$

where,  
 $F_k$  : force of the connection spring  
 $F_c$  : force of the dashpot and spring  
 $F_{ij}$  : force between i-j nodes

The force of the connection spring,  $F_k$ , is obtained as,

$$F_k = K_d u_k = K_d (u_{ij} - u_c) \quad (3-7-27)$$

where,  
 $u_k$  : relative displacement of the connection spring  
 $u_c$  : relative displacement of the dashpot and spring  
 $u_{ij}$  : relative displacement between i-j nodes

The force of the dashpot and spring is,

$$F_c = K u_c + C \dot{u}_c \quad (3-7-28)$$

Substituting Equations (3-7-27) and (3-7-28) into (3-7-26)

$$K_d (u_{ij} - u_c) = K u_c + C \dot{u}_c \quad (3-7-29)$$

When the time interval  $\Delta t$  is small enough, the velocity at time  $t$  can be expressed as,

$$\dot{u}_c(t) = \frac{\Delta u_c(t)}{\Delta t} \quad (3-7-30)$$

$$\Delta u_c(t) = u_c(t) - u_c(t - \Delta t) \quad (3-7-31)$$

Substituting above equations into Equation (3-7-29),

$$\Delta u_c(t) = \frac{K_d u_{ij}(t) - (K_d + K) u_c(t - \Delta t)}{\frac{C}{\Delta t} + (K_d + K)} \quad (3-7-32)$$

The algorithm to obtain the force  $F_{ij}(t)$  from  $u_{ij}(t)$  is as follows:

- 1) Evaluate  $\Delta u_c(t)$  from Equation (3-7-32)
- 2) Evaluate  $u_c(t)$  from Equation (3-7-30)
- 3) Evaluate  $F_{ij}(t)$  from Equation (3-7-27)

In case there is no elastic spring,

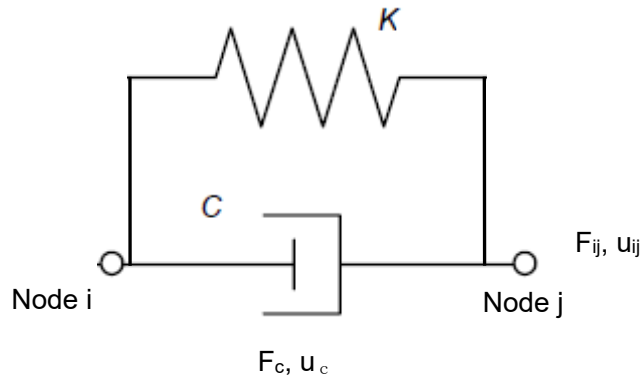


Figure 3-7-11 Voigt model without connection spring

$$u_{ij}(t) = u_c(t)$$

$$F_{ij} = F_c = K u_c + C \dot{u}_c$$

$$\dot{u}_c(t) = \frac{\Delta u_c(t)}{\Delta t} = \frac{\Delta u_{ij}(t)}{\Delta t}$$

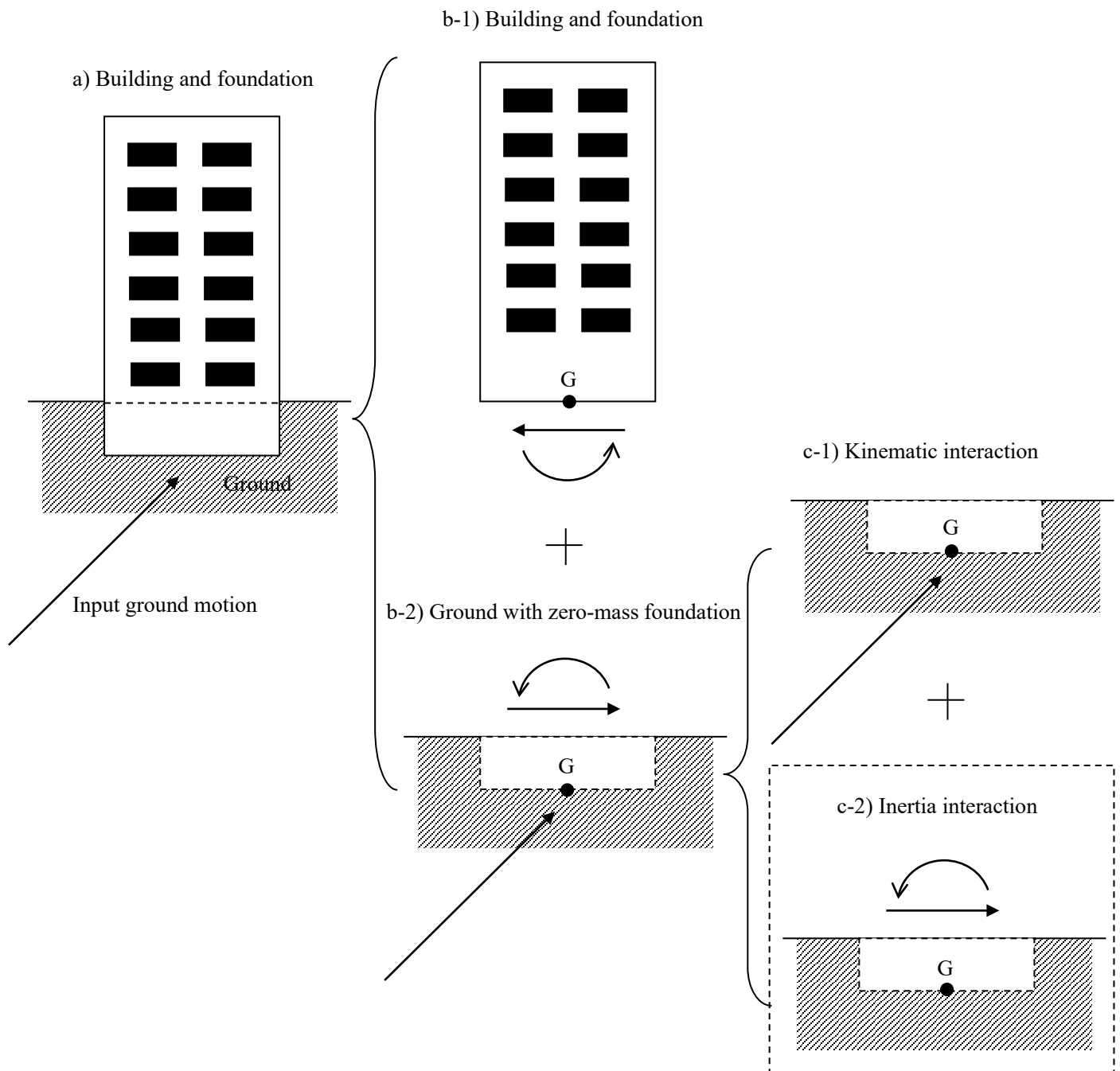
Therefore,

$$F_{ij}(t) = K u_{ij}(t) + C \frac{\Delta u_{ij}(t)}{\Delta t} \quad (3-7-33)$$

### 3.8 Ground Spring

#### 3.8.1 Soil structure interaction

a) When building and foundation on ground are subjected to an earthquake excitation, the system can be divided into two parts: b-1) building and foundation with interaction forces and b-2) ground with zero-mass foundation subjected to the reaction of interaction forces and an earthquake excitation, which can be divided further into c-1) zero-mass foundation subjected to an earthquake excitation (**kinematic interaction**) and c-2) zero-mass foundation subjected to the reaction of interaction forces (**inertia interaction**).



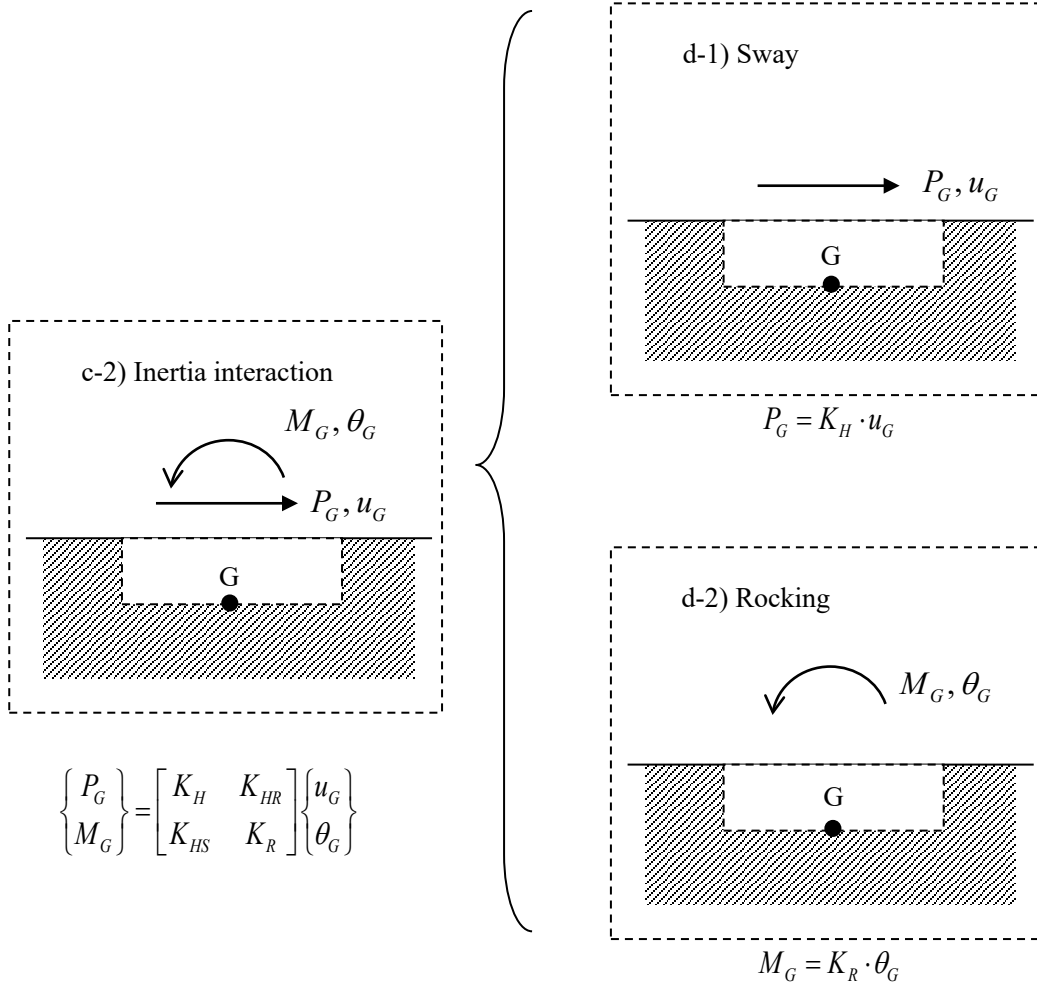
In case of c-2), the force-displacement relationship is written as,

$$\begin{Bmatrix} P_G \\ M_G \end{Bmatrix} = \begin{bmatrix} K_H & K_{HR} \\ K_{HS} & K_R \end{bmatrix} \begin{Bmatrix} u_G \\ \theta_G \end{Bmatrix} \quad (3-8-1)$$

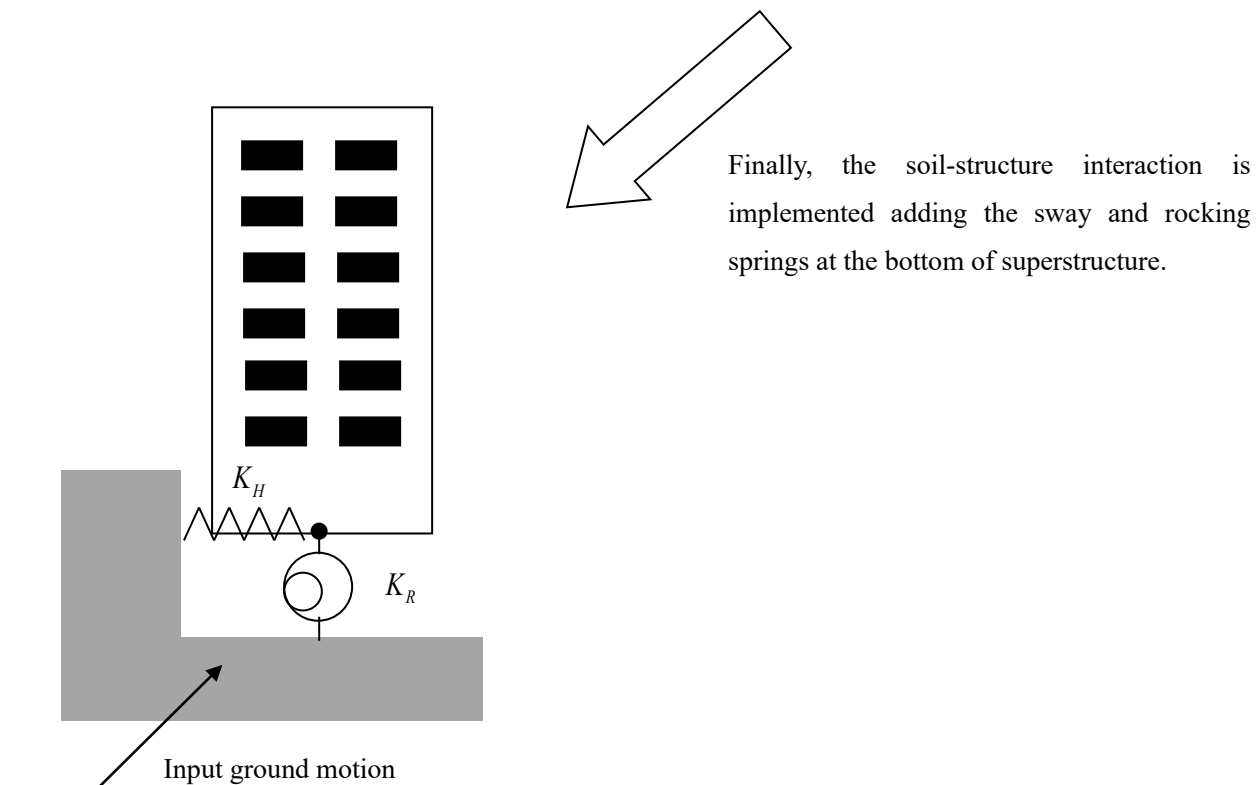
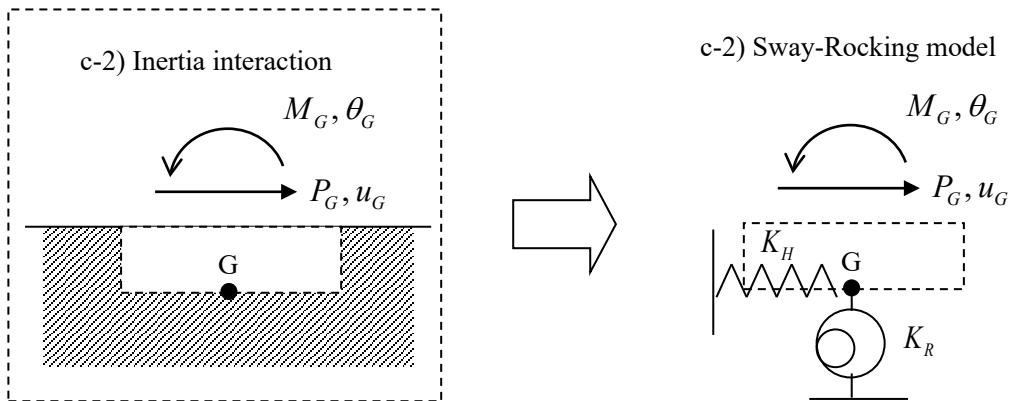
where  $P_G, M_G$  are sway and rocking forces corresponding to the interaction forces between the superstructure (building-foundation) and the ground,  $u_G, \theta_G$  are sway and rocking displacements. This stiffness matrix is called “**dynamic impedance matrix**”.

If we neglect the coupling between sway and rocking degrees of freedom, the dynamic impedance matrix is evaluated separately from the d-1) sway impedance  $K_H$  and d-2) rocking impedance  $K_R$  as follows:

$$\begin{Bmatrix} P_G \\ M_G \end{Bmatrix} = \begin{bmatrix} K_H & 0 \\ 0 & K_R \end{bmatrix} \begin{Bmatrix} u_G \\ \theta_G \end{Bmatrix} \quad (3-8-2)$$



This corresponds to the **Sway-Rocking model** as shown below:



It is important to note that the input ground motion to an embedded foundation is smaller than the input ground motion in the free field due to the influence of the embedding of the foundation. This effect is called “**kinematic interaction**”.



### 3.8.2 Cone model to calculate the static stiffness

The cone model is proposed by Wolf [1994] for determining the dynamic stiffness of a foundation on the ground. The foundation is assumed as an equivalent rigid cylinder and only vertically incident shear wave is considered. In case of the stratified ground, a simplified formulation is proposed by Iiba et.al. [2000] without considering the reflection and refraction coefficients at the boundary of the soil layer to obtain the static stiffness. The following formulation is adopted in the STERA\_3D software.

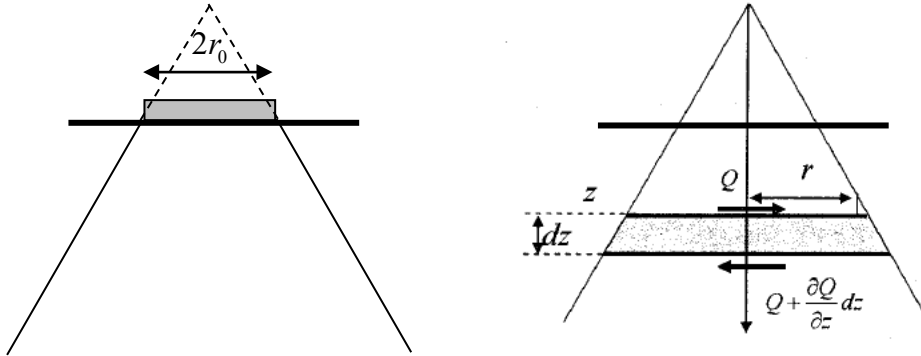
#### Reference:

- 1) John P Wolf, Foundation Vibration Analysis Using Simple Physical Models, Prentice Hall, 1994
- 2) Iiba M., Miura K and Koyamada K, "Simplified Method for Static Soil Stiffness of Surface Foundation", Proceedings of AIJ Annual Meeting, 303-304, AIJ, 2000. (in Japanese)

#### a) Sway spring

Consider a semi-infinite cone whose area increases in the depth direction. First, we show the calculation method of the horizontal ground spring (sway spring) for the rectangular foundation  $2b \times 2c$  (ground surface foundation or embedded foundation). The equivalent radius of a circle having the same area is

obtained as  $r_0 = 2\sqrt{\frac{bc}{\pi}}$ .



The forces of the minute portion at the distance  $z$  from the apex of the cone are:

- Shear force at the upper surface

$$Q = \pi r^2 G \gamma = \pi r^2 G \frac{\partial u}{\partial z} \quad (3-8-3)$$

- Shear force at the lower surface

$$Q + \frac{dQ}{dz} dz = \pi \left( \left( 1 + \frac{dz}{z} \right) r \right)^2 G \frac{\partial}{\partial z} \left( u + \frac{\partial u}{\partial z} dz \right) = \pi \left( 1 + \frac{dz}{z} \right)^2 r^2 G \left( \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2} dz \right) \quad (3-8-4)$$

Considering the static case ignoring the inertial force acting on the minute part, from the balancing of forces,

$$\rightarrow \left( Q + \frac{dQ}{dz} dz \right) - Q = 0$$

$$\rightarrow \pi \left( 1 + \frac{dz}{z} \right)^2 r^2 G \left( \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2} dz \right) - \pi r^2 G \frac{\partial u}{\partial z} = 0$$

$$\rightarrow \left( 1 + \frac{dz}{z} \right)^2 \left( \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2} dz \right) - \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial z^2} dz + \left( 2 \frac{dz}{z} + \left( \frac{dz}{z} \right)^2 \right) \left( \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2} dz \right) = 0$$

→ Ignoring high-order small amount terms

$$\frac{\partial^2 u}{\partial z^2} + \frac{2}{z} \frac{\partial u}{\partial z} = 0 \quad (3-8-5)$$

The solution to this equation can be expressed as follows:

$$u = \frac{A}{z} + B \quad (3-8-6)$$

where A and B as undetermined coefficients.

Assuming that the displacement on the ground surface is U and the displacement at the depth d is 0 as boundary conditions,

$$U = \frac{A}{l} + B, \quad 0 = \frac{A}{d} + B \quad (3-8-7)$$

From this, the coefficient A is

$$A = \frac{(l+d)l}{d} U \quad (3-8-8)$$

Let  $Q_0$  be the shear force of the ground surface

$$Q_0 = \pi r_0^2 G \frac{\partial u}{\partial z} = \pi r_0^2 G \left( -\frac{A}{l^2} \right) = - \left( \pi r_0^2 G \frac{l+d}{ld} \right) U \quad (3-8-9)$$

Therefore, the horizontal spring  $K_H$  on the ground surface is

$$K_H = -\frac{Q_0}{U} = \pi r_0^2 G \frac{l+d}{ld} \quad (3-8-10)$$

Assuming that d is infinite,

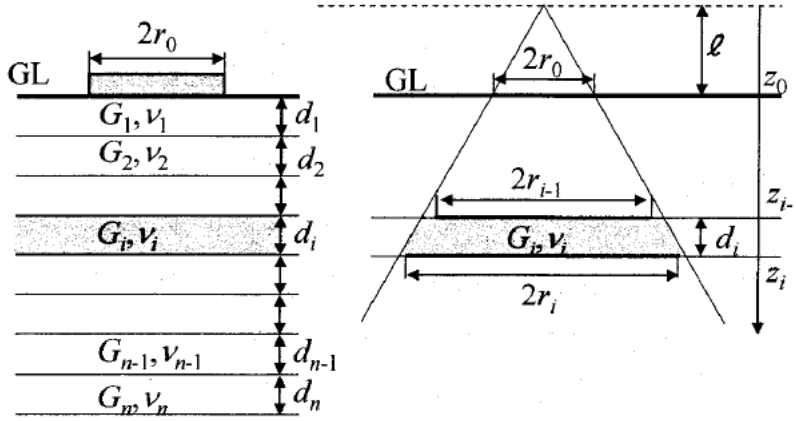
$$K_H = \frac{\pi r_0^2 G}{l} \quad (3-8-11)$$

The horizontal spring of the circular rigid foundation on semi-infinite uniform ground is obtained theoretically from the following formula.

$$K_H = \frac{8Gr_0}{2-\nu} \quad (3-8-12)$$

If the two springs are set to be equal, the distance  $l$  from the apex of the cone to the ground surface is obtained as follows:

$$\frac{8Gr_0}{2-\nu} = \frac{\pi r_0^2 G}{l} \rightarrow l = \frac{(2-\nu)}{8} \pi r_0 \quad (3-8-13)$$



In case of the stratified ground, consider a truncated cone of thickness  $d_i$  from the  $i$ -th layer of stratified ground and  $z_i$  be the coordinate of the bottom of the  $i$ -th layer. The radius of the truncated cone  $r_i$  at depth  $z_i$  is then calculated as follows from the geometric relationship.

$$r_i = \frac{z_i}{z_0} r_0 \quad (3-8-14)$$

The horizontal spring on the upper surface of this truncated cone is

$$K_H^i = \pi r_{i-1}^2 G_i \frac{z_{i-1} + d_i}{z_{i-1} d_i} = \pi \left( \frac{z_{i-1}}{z_0} r_0 \right)^2 G_i \frac{z_i}{z_{i-1} (z_i - z_{i-1})} = \frac{\pi r_0^2 G_1}{z_0} \left( \frac{G_i}{G_1} \right) \frac{z_i z_{i-1}}{z_0 (z_i - z_{i-1})} \quad (3-8-15)$$

The horizontal spring  $K_{hb}$  at the base bottom position is obtained as a synthetic spring in which horizontal springs of each layer are connected in series.

$$\frac{1}{K_{hb}} = \sum_{i=0}^{n-1} \frac{1}{K_H^i} \quad (3-8-16)$$

However, in the bottom layer,

$$K_H^{n-1} = \frac{\pi r_0^2 G_1}{z_0} \left( \frac{G_n}{G_1} \right) \frac{z_n z_{n-1}}{z_0 (z_n - z_{n-1})} \rightarrow \frac{\pi r_0^2 G_1}{z_0} \left( \frac{G_n}{G_1} \right) \frac{z_{n-1}}{z_0} \quad (z_n \rightarrow \infty) \quad (3-8-17)$$

Finally, the horizontal ground spring  $K_{hb}$  is obtained as,

$$K_{hb} = \beta_h K_{1h} \quad (3-8-18)$$

where

$$\beta_h = \frac{1}{\sum_{i=1}^n \left( \frac{1}{\alpha_i} \right)}, \quad K_{1h} = \frac{\pi r_0^2 G_1}{z_0} = \frac{\pi r_0^2 G_1}{l} = \frac{8\pi r_0 G_1}{(2-\nu)}$$

$$\alpha_i = \left( \frac{G_i}{G_1} \right) \frac{z_i z_{i-1}}{z_0 (z_i - z_{i-1})} \quad (i=1, 2, \dots, n-1), \quad \alpha_n = \left( \frac{G_n}{G_1} \right) \frac{z_{n-1}}{z_0}$$

$$z_0 = \frac{2-\nu_1}{8} \pi r_0$$

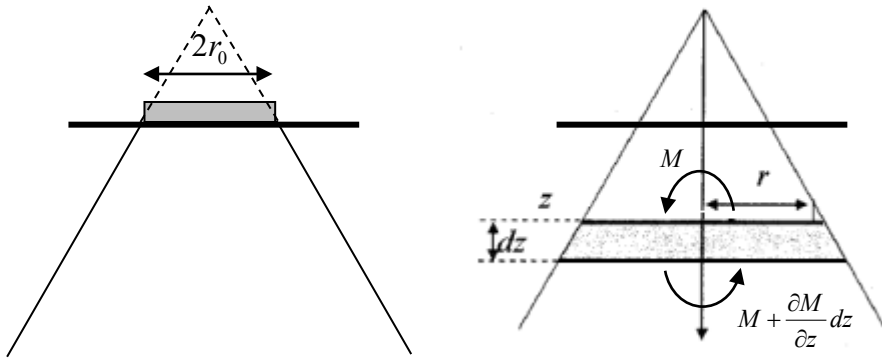
### b) Rocking spring

Rotational spring can be obtained as follows, similar to the method for determining horizontal spring. For the rectangular foundation  $2b \times 2c$  (ground surface foundation or embedded foundation,  $2b$  is the length in rotational direction), the equivalent radius of a circle having the same moment of inertia is

obtained as  $r_{r0} = \sqrt[4]{\frac{(2b)^3 (2c)}{3\pi}}$ .

$\therefore$ ) The moment of inertia of a circle  $I_c = \frac{\pi}{4} r_{r0}^4$

The moment of inertia of a rectangular  $I_b = \frac{(2b)^3 (2c)}{12}$



The forces of the minute portion at the distance  $z$  from the apex of the cone are:

- Moment at the upper surface

$$M = -EI \frac{\partial \theta}{\partial z} = E \frac{\pi r_{r0}^4}{4} \frac{\partial \theta}{\partial z} \quad (3-8-19)$$

- Moment at the lower surface

$$M + \frac{dM}{dz} dz = -\frac{\pi}{4} \left( \left( 1 + \frac{dz}{z} \right) r_{r0} \right)^4 E \frac{\partial}{\partial z} \left( \theta + \frac{\partial \theta}{\partial z} dz \right) = -\frac{\pi}{4} \left( 1 + \frac{dz}{z} \right)^4 r_{r0}^4 E \left( \frac{\partial \theta}{\partial z} + \frac{\partial^2 \theta}{\partial z^2} dz \right) \quad (3-8-20)$$

Considering the static case ignoring the inertial force acting on the minute part, from the balancing of forces,

$$\begin{aligned} \rightarrow & \left( M + \frac{dM}{dz} dz \right) - M = 0 \\ \rightarrow & -\frac{\pi}{4} \left( 1 + \frac{dz}{z} \right)^4 r_{r0}^4 E \left( \frac{\partial \theta}{\partial z} + \frac{\partial^2 \theta}{\partial z^2} dz \right) + \frac{r_{r0}^4}{4} E \frac{\partial \theta}{\partial z} = 0 \\ \rightarrow & \text{Ignoring high-order small amount terms} \end{aligned}$$

$$\frac{\partial^2 \theta}{\partial z^2} + \frac{4}{z} \frac{\partial \theta}{\partial z} = 0 \quad (3-8-21)$$

The solution to this equation can be expressed as follows:

$$\theta = \frac{A}{z^3} + B \quad (3-8-22)$$

where A and B as undetermined coefficients.

Assuming that the rotational displacement on the ground surface is  $\Theta$  and the displacement at the depth d is 0 as boundary conditions,

$$\Theta = \frac{A}{l_r^3} + B, \quad 0 = \frac{A}{(l_r + d)^3} + B \quad (3-8-23)$$

From this, the coefficient A is

$$A = \frac{(l_r + d)^3 l_r^3}{(l_r + d)^3 - l_r^3} \Theta \quad (3-8-24)$$

Let  $M_0$  be the shear force of the ground surface

$$M_0 = -\frac{\pi r_{r0}^4}{4} E \frac{\partial u}{\partial z} = -\frac{\pi r_{r0}^4}{4} E \left( -\frac{3A}{l_r^4} \right) = \frac{\pi r_{r0}^4}{4} E \frac{3(l_r + d)^3}{\{(l_r + d)^3 - l_r^3\} l_r} \Theta \quad (3-8-25)$$

Therefore, the rotational spring  $K_R$  on the ground surface is

$$K_R = \frac{M_0}{\Theta} = \frac{\pi r_{r0}^4}{4} E \frac{3(l_r + d)^3}{\{(l_r + d)^3 - l_r^3\} l_r} \quad (3-8-26)$$

Assuming that d is infinite,

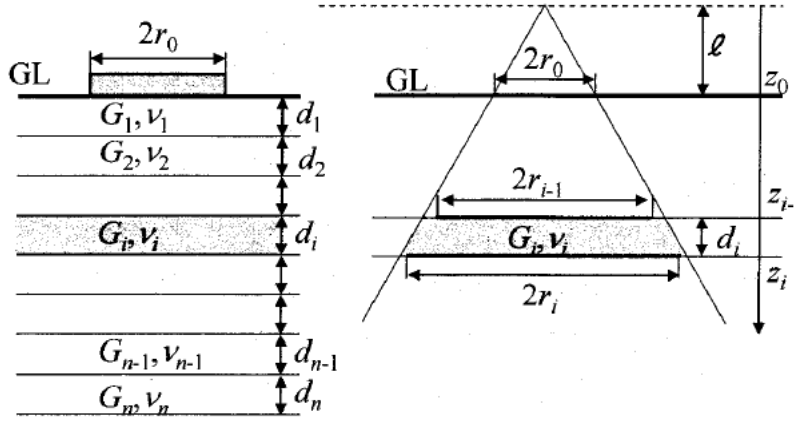
$$K_R = \frac{3\pi r_{r0}^4 E}{4l_r} \quad (3-8-27)$$

The horizontal spring of the circular rigid foundation on semi-infinite uniform ground is obtained theoretically from the following formula.

$$K_R = \frac{8Gr_{r0}^3}{3(1-\nu)} \quad (3-8-28)$$

If the two springs are set to be equal, the distance  $l_r$  from the apex of the cone to the ground surface is obtained as follows:

$$\frac{8Gr_{r0}^3}{3(1-\nu)} = \frac{3\pi r_{r0}^4 E}{4l_r} = \frac{3\pi r_{r0}^4 E}{4l_r} 2G(1+\nu) \rightarrow l_r = \frac{9(1-\nu^2)}{16} \pi r_{r0} \quad (3-8-29)$$



In case of the stratified ground, consider a truncated cone of thickness  $d_i$  from the  $i$ -th layer of stratified ground and  $z_{ri}$  be the coordinate of the bottom of the  $i$ -th layer. The radius of the truncated cone  $r_{ri}$  at depth  $z_{ri}$  is then calculated as follows from the geometric relationship.

$$r_{ri} = \frac{z_{ri}}{z_{r0}} r_{r0} \quad (3-8-30)$$

The rotational spring on the upper surface of this truncated cone is

$$K_R^i = \frac{\pi r_{ri-1}^4 E_i}{4} \frac{3(z_{ri-1} + d_i)^3}{\{(z_{ri-1} + d_i)^3 - z_{ri-1}^3\} z_{ri-1}} = \frac{3\pi r_{r0}^4 E_1}{4z_{r0}} \left( \frac{E_i}{E_1} \right) \frac{z_{ri}^3 z_{ri-1}^3}{z_{r0}^3 (z_{ri}^3 - z_{ri-1}^3)} \quad (3-8-31)$$

The rotational spring  $K_{rb}$  at the base bottom position is obtained as a synthetic spring in which rotational springs of each layer are connected in series.

$$\frac{1}{K_{rb}} = \sum_{i=0}^{n-1} \frac{1}{K_R^i} \quad (3-8-32)$$

However, in the bottom layer,

$$K_R^{n-1} = \frac{3\pi r_{r0}^4 E_1}{4z_{r0}} \left( \frac{E_n}{E_1} \right) \frac{z_{rn}^3 z_{rn-1}^3}{z_{r0}^3 (z_{rn}^3 - z_{rn-1}^3)} \rightarrow \frac{3\pi r_{r0}^4 E_1}{4z_{r0}} \left( \frac{E_n}{E_1} \right) \frac{z_{rn-1}^3}{z_{r0}^3} \quad (z_n \rightarrow \infty) \quad (3-8-33)$$

Finally, the horizontal ground spring  $K_{rb}$  is obtained as,

$$K_{rb} = \beta_r K_{1r} \quad (3-8-34)$$

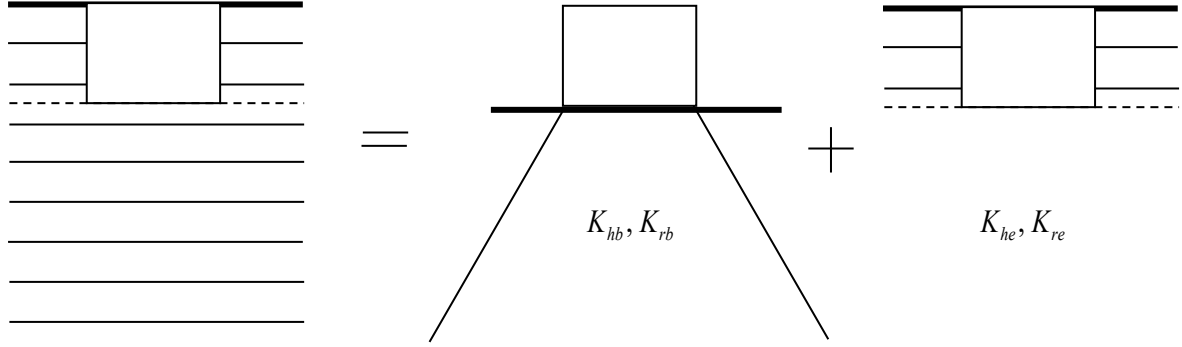
ここに、

$$\beta_r = \frac{1}{\sum_{i=1}^n \left( \frac{1}{\alpha_{ri}} \right)}, \quad K_{1r} = \frac{3\pi r_{r0}^4 E_1}{4z_{r0}} = \frac{3\pi r_{r0}^4 E_1}{4l_r} = \frac{4}{3} \frac{r_{r0}^3 E_1}{1 - \nu_1^2}$$

$$\alpha_i = \left( \frac{E_i}{E_1} \right) \frac{z_{ri}^3 z_{ri-1}^3}{z_{r0}^3 (z_{ri}^3 - z_{ri-1}^3)} \quad (i = 1, 2, \dots, n-1), \quad \alpha_n = \left( \frac{E_n}{E_1} \right) \left( \frac{z_{n-1}}{z_0} \right)^3$$

$$z_{r0} = \frac{9(1 - \nu_1^2)}{16} \pi r_{r0}$$

### 3.8.3 Embedded foundation



In case of embedded spread foundation, the resistances at the side of the foundation  $K_{he}, K_{re}$  can be expected in addition to the resistances  $K_{hb}, K_{rb}$  at the base of the foundation. That is,

$$\begin{aligned} K_h &= K_{hb} + K_{he} \\ K_r &= K_{rb} + K_{re} \end{aligned} \quad (3-8-35)$$

where

$$K_{he} = \xi_{he} K_{hb} \frac{D_e}{r_0} \frac{G_{he}}{G_{hb}} \quad (3-8-36)$$

$$K_{re} = \xi_{re} K_{rb} \left\{ 2.3 \frac{D_e}{r_0} + 0.58 \left( \frac{D_e}{r_0} \right)^3 \right\} \frac{G_{he}}{G_{hb}} \quad (3-8-37)$$

$$G_{he} = \frac{\sum_{i=1}^m G_i H_i}{\sum_{i=1}^m H_i}, \quad G_{hb} = \frac{(2-\nu) K_{hb}}{8r_0} \quad (3-8-38)$$

$D_e$  is the depth of the foundation.  $\xi_{he}$  and  $\xi_{re}$  are the earth pressure reduction coefficients of horizontal and rotational directions at the side of the foundation and they are set to 0.5 when considering only the side receiving the reaction force from ground at the time of the earthquake.  $m$  is the number of soil layers from the surface to the bottom at the side the foundation where the earth pressure acts.  $\nu$  is the average Poisson's ratio of the ground under the foundation base. The damping at the embedded part is not considered.

### 3.8.4 Radiation damping

The static stiffness obtained by the cone model alone can not express the radiation damping that the energy of ground shaking spreads to a distance.

To evaluate the radiation damping, we consider a semi-infinite earth column with the same area of the foundation where a shear wave travels downward when the foundation sways harmonically in a horizontal direction.

The wave travels in the earth column can be expressed as the solution of the wave equation.

$$\frac{\partial^2 u}{\partial t^2} = V_s^2 \frac{\partial^2 u}{\partial z^2}, \quad V_s = \sqrt{\frac{G}{\rho}} \quad (3-8-39)$$

where  $G$  is the shear modulus of the soil,  $\rho$  is the density of the soil, and  $V_s$  is the shear wave velocity.

When the foundation sways harmonically as  $ue^{ipt}$ , the solution of the wave equation is

$$u(z, t) = ue^{ip(t-z/V_s)} \quad (3-8-40)$$

The shear force at the bottom of the foundation is,

$$Q = -GA \frac{\partial u}{\partial z} \Big|_{z=0} = \frac{GA}{V_s} iupe^{ipt} = \frac{GA}{V_s} \frac{du}{dt} = (\rho V_s A) \frac{du}{dt} \quad (3-8-41)$$

where  $A$  is the area of the foundation. Therefore, the damping force by the radiation is equivalent as the viscous damping of a dashpot with a damping coefficient

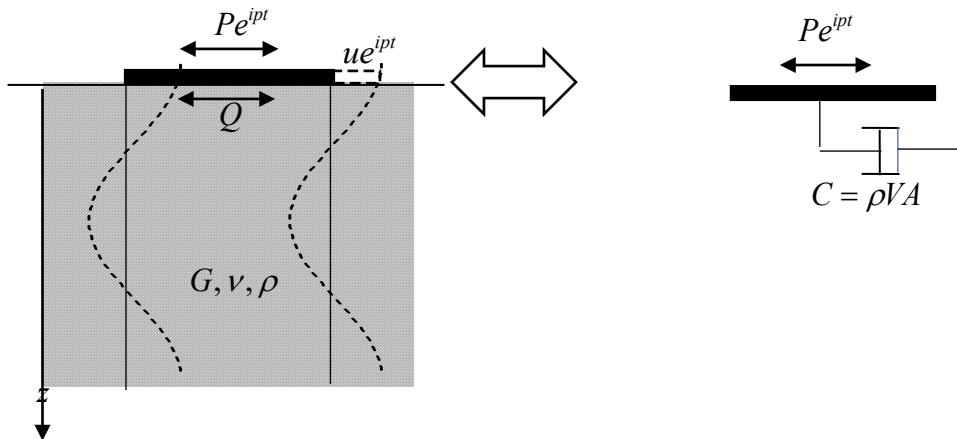
$$C_H = \rho V_s A \quad (3-8-42)$$

The radiation damping of a rocking motion is expressed as the similar formula

$$C_R = \rho(\eta V_s) I \quad (3-8-43)$$

where  $I = \frac{\pi r^4}{4}$  is the second moment of inertia for a circular foundation with the radius  $r$

$\eta = \frac{3.4}{\pi(1-\nu)}$  is the coefficient for vertical wave velocity, where  $\nu$  is the Poisson's ratio



In case of the stratified ground, we can use the following formula for the radiation damping

$$C_H = \rho_e V_e A \quad (3-8-44)$$

$$C_R = \rho_e (\eta_e V_e) I, \quad \eta_e = \frac{3.4}{\pi(1-\nu_e)} \quad (3-8-45)$$

where  $\rho_e$  is the average density,  $V_e$  is the average shear wave velocity and  $\nu_e$  is the average shear modulus defined by the weighted average by depth of layers under the basement as

$$\rho_e = \frac{\sum_{i=1}^n \rho_i d_i}{\sum_{i=1}^n d_i}, \quad V_e = \frac{\sum_{i=1}^n V_i d_i}{\sum_{i=1}^n d_i}, \quad \nu_e = \frac{\sum_{i=1}^n \nu_i d_i}{\sum_{i=1}^n d_i} \quad (3-8-46)$$

### 3.8.5 Complex stiffness with material damping

The damping effect of the soil material can be considered by setting the shear modulus to the following complex shear modulus.

$$G^* = G(1 + 2ih) \quad (3-8-47)$$

where  $h$  is the damping factor of the soil. As a result, the dynamic stiffness obtained from the cone model becomes also complex value as,

$$\begin{aligned} K_H^* &= K_H + iK_H' = K_H(1 + 2ih_H) \quad : \text{sway spring} \\ K_R^* &= K_R + iK_R' = K_R(1 + 2ih_R) \quad : \text{rocking spring} \end{aligned} \quad (3-8-48)$$

Furthermore, the damping coefficient is obtained from the imaginary part of the complex stiffness under the periodic vibration of the circular frequency  $\omega$ .

$$K \cdot x + C \cdot \dot{x} \rightarrow \text{assuming } x = ae^{i\omega t} \rightarrow (K + i\omega C)x$$

From the equivalent condition,

$$(K + i\omega C)x = (K + iK')x = K(1 + 2ih)x$$

$$\text{Therefore,} \quad C = \frac{K'}{\omega} = \frac{2hK}{\omega} \quad (3-8-49)$$

STERA\_3D calculates the circular frequency  $\omega$  as

$$\omega_1 = \frac{2\pi}{T_1} \quad (3-8-50)$$

where  $T_1$  is the first natural period of the structure with the ground spring (real part).

### 3.8.6 Impedance matrix

It is known that radiation damping is likely to occur in a frequency band higher than the dominant frequency of the ground ( $f_G$ ), and the effect is greater at higher frequencies. Therefore, the damping is evaluated separately for a lower frequency side and a higher frequency side than the dominant frequency.

- a) In case of  $f \leq f_G$  ( $\omega \leq \omega_G$ ) for Sway spring and  $f \leq 2f_G$  ( $\omega \leq 2\omega_G$ ) for Rocking spring

Considering material damping only,

$$P_G = K_H u_G + C_H \dot{u}_G, \quad C = \frac{K'}{\omega} = \frac{2hK}{\omega} \quad (3-8-51)$$

$$M_G = K_R \theta_G + C_R \dot{\theta}_G \quad (3-8-52)$$

where

$K_H, C_H$  : stiffness and damping of sway spring

$K_R, C_R$  : stiffness and damping of rocking spring

- b) In case of  $f > f_G$  ( $\omega > \omega_G$ ) for Sway spring and  $f > 2f_G$  ( $\omega > 2\omega_G$ ) for Rocking spring

Considering both material damping and radiation damping,

$$P_G = K_H u_G + (C_H + C_H') \dot{u}_G \quad (3-8-53)$$

$$M_G = K_R \theta_G + (C_R + C_R') \dot{\theta}_G \quad (3-8-54)$$

where

$C_H', C_R'$  : radiation damping for sway and rocking

To avoid the discontinuous of damping, we modify the formula as

$$P_G = K_H u_G + (C_H + \zeta_H C_H') \dot{u}_G, \quad \zeta_H = \frac{f - f_G}{f} \quad (3-8-55)$$

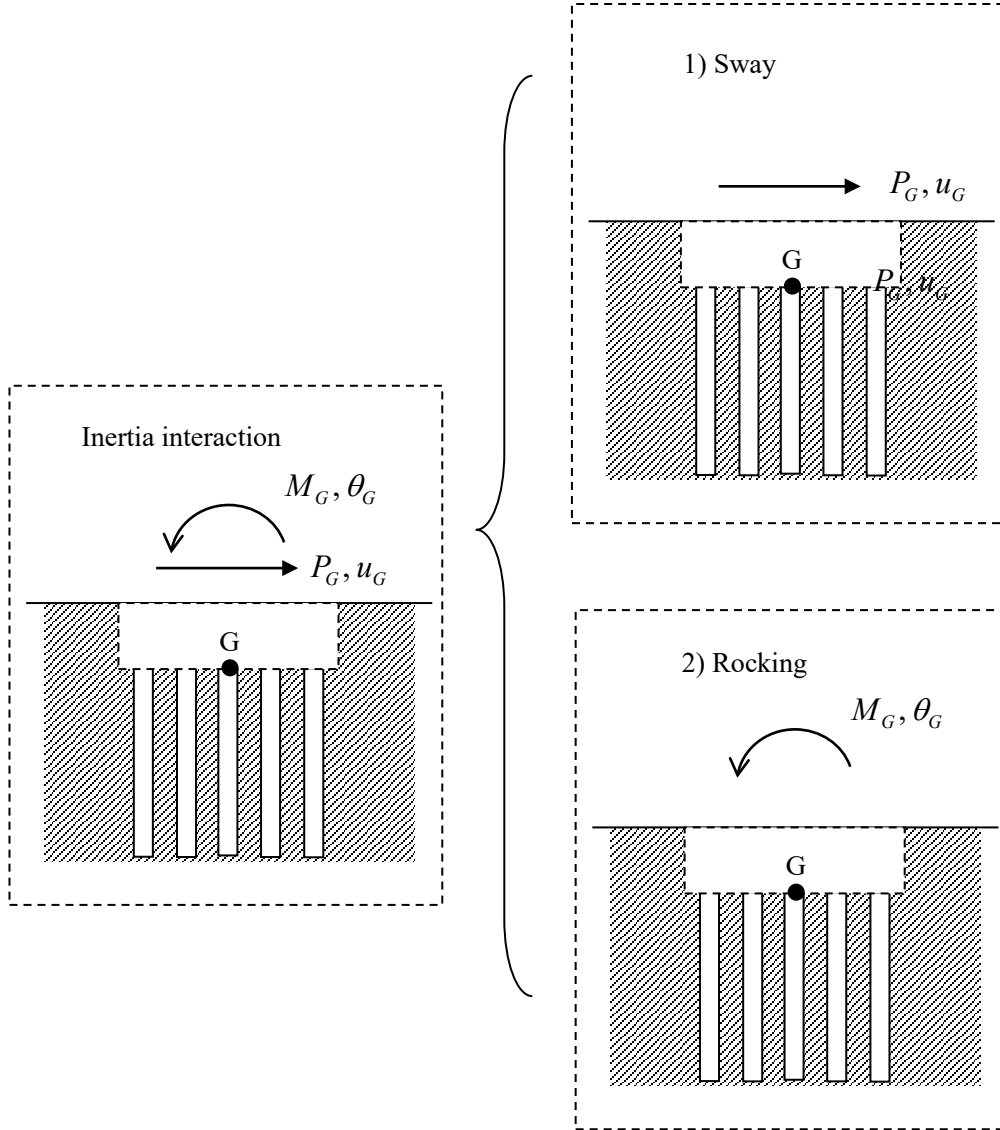
$$M_G = K_R \theta_G + (C_R + \zeta_R C_R') \dot{\theta}_G, \quad \zeta_R = \frac{f - 2f_G}{f} \quad (3-8-56)$$

In a matrix form

$$\begin{Bmatrix} P_G \\ M_G \end{Bmatrix} = \begin{bmatrix} K_H & 0 \\ 0 & K_R \end{bmatrix} \begin{Bmatrix} u_G \\ \theta_G \end{Bmatrix} + \begin{bmatrix} C_H + \zeta_H C_H' & 0 \\ 0 & C_R + \zeta_R C_R' \end{bmatrix} \begin{Bmatrix} \dot{u}_G \\ \dot{\theta}_G \end{Bmatrix} \quad (3-8-57)$$

### 3.8.7 Pile foundation

Now we discuss the Sway and Rocking springs for the foundation with piles.



#### a) Vertical stiffness of a single pile

The vertical stiffness of a single pile is obtained from the follow formula:

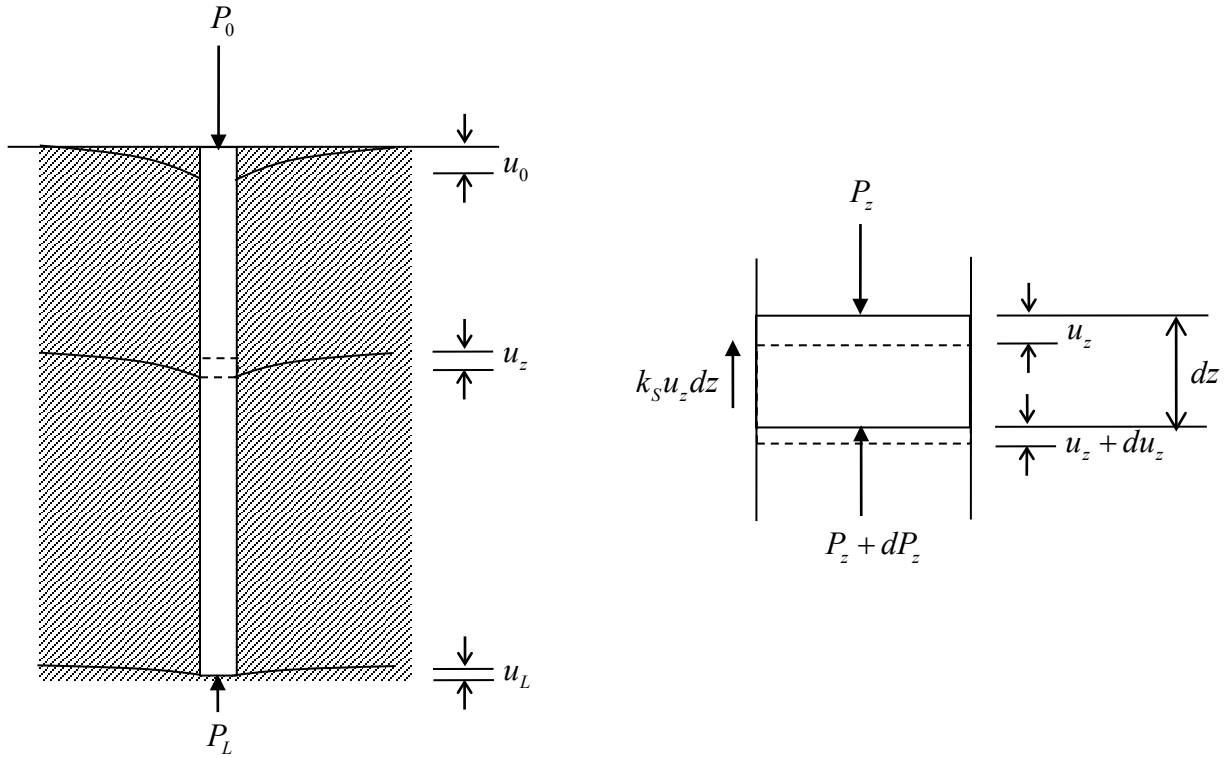
$$K_V = EA\alpha \frac{k_B(1 + e^{-2\alpha L}) + EA\alpha(1 - e^{-2\alpha L})}{k_B(1 - e^{-2\alpha L}) + EA\alpha(1 + e^{-2\alpha L})}, \quad \alpha = \sqrt{\frac{k_S}{EA}} \quad (3-8-58)$$

where,

$E$  : Young's Modulus of the pile,  $A$  : Area of the pile,  $L$  : Length of the pile

$k_S$  : Vertical spring of the soil surrounding the pile,  $k_B$  : Vertical spring at the bottom of the pile

∴)



#### a-1) Equilibrium condition of the vertical forces in a pile

The equilibrium condition of the vertical forces in a small segment is

$$dP = k_s u_z dz = 0 \quad (3-8-59)$$

The axial strain in the segment is obtained as

$$\frac{du_z}{dz} = -\frac{P_z}{EA} \quad (3-8-60)$$

Therefore

$$\frac{dP}{dz} = EA \frac{d^2 u_z}{dz^2} = k_s u_z \quad (3-8-61)$$

The solution of this second order differential equation is

$$u_z = c_1 e^{\alpha z} + c_2 e^{-\alpha z}, \quad \alpha = \sqrt{\frac{k_s}{EA}} \quad (3-8-62)$$

Also

$$P_z = EA \alpha (c_2 e^{-\alpha z} - c_1 e^{\alpha z}) \quad (3-8-63)$$

Setting the boundary conditions as  $P_z = P_0$  at  $z = 0$  and  $u_z = u_L$  at  $z = L$ ,

$$P_0 = EA \alpha (c_2 - c_1) \quad (3-8-64)$$

$$u_L = c_1 e^{\alpha L} + c_2 e^{-\alpha L} \quad (3-8-65)$$

Therefore, the coefficients  $c_1$  and  $c_2$  are obtained as

$$c_1 = \frac{EA\alpha u_L - P_0 e^{-\alpha L}}{EA\alpha(e^{\alpha L} + e^{-\alpha L})}, \quad c_2 = \frac{EA\alpha u_L + P_0 e^{\alpha L}}{EA\alpha(e^{\alpha L} + e^{-\alpha L})} \quad (3-8-66)$$

The force at the bottom of the pile  $P_L$  is

$$P_L = EA\alpha(c_2 e^{-\alpha L} - c_1 e^{\alpha L}) \quad (3-8-67)$$

From the relationship  $u_L = P_L/k_B$ ,

$$P_L = \frac{2K_P P_0}{K_P(e^{\alpha L} + e^{-\alpha L}) + EA\alpha(e^{\alpha L} - e^{-\alpha L})} \quad (3-8-68)$$

and

$$u_L = \frac{2P_0}{K_P(e^{\alpha L} + e^{-\alpha L}) + EA\alpha(e^{\alpha L} - e^{-\alpha L})} \quad (3-8-69)$$

The displacement at the head of the pile is

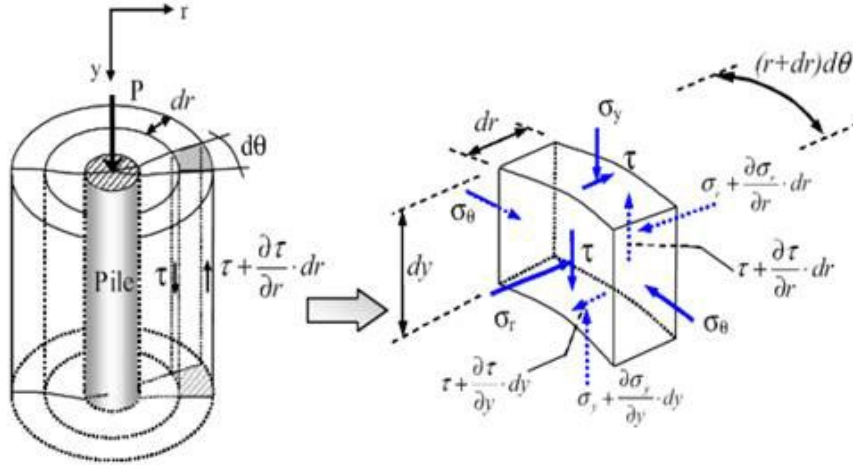
$$u_0 = c_1 + c_2 \quad (3-8-68)$$

Therefore, the stiffness of the vertical spring at the head of the pile is

$$\begin{aligned} K_0 &= \frac{P_0}{u_0} = \frac{EA\alpha(c_2 - c_1)}{c_1 + c_2} = \frac{EA\alpha(P_0 e^{\alpha L} + P_0 e^{-\alpha L})}{2EA\alpha u_L - P_0 e^{-\alpha L} + P_0 e^{\alpha L}} \\ &= \frac{EA\alpha(e^{\alpha L} + e^{-\alpha L})}{2EA\alpha \frac{2}{k_B(e^{\alpha L} + e^{-\alpha L}) + EA\alpha(e^{\alpha L} - e^{-\alpha L})} + (e^{\alpha L} - e^{-\alpha L})} \\ &= \frac{EA\alpha(e^{\alpha L} + e^{-\alpha L})\{k_B(e^{\alpha L} + e^{-\alpha L}) + EA\alpha(e^{\alpha L} - e^{-\alpha L})\}}{4EA\alpha + (e^{\alpha L} - e^{-\alpha L})\{k_B(e^{\alpha L} + e^{-\alpha L}) + EA\alpha(e^{\alpha L} - e^{-\alpha L})\}} \\ &= EA\alpha \frac{k_B(e^{\alpha L} + e^{-\alpha L}) + EA\alpha(e^{\alpha L} - e^{-\alpha L})}{k_B(e^{\alpha L} - e^{-\alpha L}) + EA\alpha(e^{\alpha L} + e^{-\alpha L})} \end{aligned} \quad (3-8-69)$$

### a-2) Vertical spring of the soil surrounding the pile

The vertical spring of the soil surrounding the pile  $k_s$  is obtained as the friction resistance of soil surrounding the soil (Randolph and Wroth, 1978).



(a) Concentric cylinder around loaded pile (b) Stresses in soil element

**Reference:** Randolph M.F and Wroth C.P, “Analysis and deformation of vertically loaded piles”, Journal of Geotechnical Engineering 104(12): 1465-1487. 1978.

From the equilibrium condition of vertical forces

$$\left( \tau + \frac{\partial \tau}{\partial r} \right) (r + dr) d\theta dy - \tau r d\theta dy + \left( \sigma_y + \frac{\partial \sigma_y}{\partial y} dy \right) \left( r + \frac{dr}{2} \right) d\theta dr - \sigma_y \left( r + \frac{dr}{2} \right) d\theta dr = 0 \quad (3-8-70)$$

Neglecting higher order

$$\frac{\partial(\tau r)}{\partial r} + r \frac{\partial \sigma_y}{\partial y} = 0 \quad (3-8-71)$$

Assuming the stress change along the depth  $\partial \sigma_y / \partial y$  is negligible, the second term will be zero. Then,

$$\frac{\partial(\tau r)}{\partial r} = 0 \quad (3-8-72)$$

Integrating from the pile radius  $r_0$  to  $r$ ,

$$\int_{r_0}^r d(\tau r) = 0 \Rightarrow \tau(r)r - \tau(r_0)r_0 = 0$$

$$\tau(r) = \frac{\tau(r_0)r_0}{r} = \frac{\tau_0 r_0}{r} \quad (3-8-73)$$

Assuming the deformation along the radius  $du$  is smaller than the deformation along the depth  $dw$ , the shear strain is

$$\gamma = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \approx \frac{dw}{dr} = \frac{\tau(r)}{G(r)} = \frac{\tau_0 r_0}{r G(r)} \quad (3-8-74)$$

The vertical shear deformation is obtained by integrating from  $r_0$  to  $r_m$ ,

$$w_s = \tau_0 r_0 \int_{r_0}^{r_m} \frac{1}{rG} dr = \frac{\tau_0 r_0}{G} \ln \left( \frac{r_m}{r_0} \right) \quad (3-8-75)$$

Randolf and Worth proposed the following empirical formula for the radius  $r_m$

$$r_m = 2.5L(1 - \nu) \quad (3-8-76)$$

The vertical force around the pile is calculated as

$$P = (2\pi r_0) \tau_0 = \left( \frac{2\pi G}{\ln(r_m/r_0)} \right) w_s \quad (3-8-77)$$

Therefore, the vertical spring of the soil surrounding the pile  $k_s$  is

$$k_s = \frac{2\pi G_e}{\ln(r_m/r_0)} \quad , \quad r_m = 2.5L(1 - \nu_e) \quad (3-8-78)$$

where,

$$G_e = \frac{1}{L} \sum_{i=1}^n (G_i d_i) : \text{average shear modulus}, \quad \nu_e = \frac{1}{L} \sum_{i=1}^n (\nu_i d_i) : \text{average Poisson ratio}$$

### a-3) Vertical spring at the bottom of the pile

The vertical spring at the bottom of the pile  $k_B$  is obtained as a static impedance of circular foundation as,

$$k_B = \frac{3\pi}{8} \frac{\pi G_B r_0}{(1 - \nu_B)} \quad (3-8-79)$$

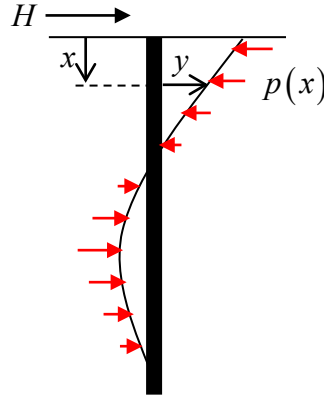
where,

$G_B$  : shear modulus of the soil at the bottom of the pile

$\nu_B$  : Poisson ratio of the soil at the bottom of the pile

## b) Horizontal stiffness of a single pile

### b-1) Horizontal stiffness of a single pile



The flexural deformation of the infinite pile under horizontal load at the top of the pile is

$$EI \frac{d^4 y}{dx^4} + p(x) = 0 \quad (3-8-80)$$

where  $p(x)$  is the reaction force of the soil.

Assuming

$$p(x) = k_h B y \quad (3-8-81)$$

where  $B$  is the width of the pile.

The solution is expressed as

$$y = e^{\beta x} (A_1 \sin \beta x + B_1 \cos \beta x) + e^{-\beta x} (C_1 \sin \beta x + D_1 \cos \beta x) \quad (3-8-82)$$

$$\beta = \sqrt[4]{\frac{k_h B}{4EI}}$$

Since the deformation in infinite depth is zero, that is  $x \rightarrow \infty$ ,  $y = 0$ ,

$$A_1 = B_1 = 0 \quad (3-8-83)$$

In case of fixed pile head,

$$\theta(0) = \left. \frac{dy}{dx} \right|_{x=0} = -\beta C_1 + \beta D_1 = 0 \rightarrow C_1 = D_1 \quad (3-8-84)$$

The horizontal force at the pile head is

$$H = -Q(0) \quad (3-8-85)$$

Therefore,

$$\frac{Q(0)}{EI} = -\left. \frac{d^3 y}{dx^3} \right|_{x=0} = -4C_1 \beta^3 = -\frac{H}{EI} \rightarrow C_1 = \frac{H}{4\beta^3 EI} \quad (3-8-86)$$

The horizontal deformation of the pile is

$$y = \frac{H}{4\beta^3 EI} e^{-\beta x} (\sin \beta x + \cos \beta x) \quad (3-8-87)$$

The deformation of the pile head is

$$y = -\frac{H}{4\beta^3 EI} \quad (3-8-88)$$

Therefore, the horizontal stiffness is

$$K_h = 4\beta^3 EI = (4EI)^{1/4} (k_h B)^{3/4} \quad (3-8-89)$$

Francis (1964) proposed the following formula for the horizontal ground spring per unit length of a single pile:

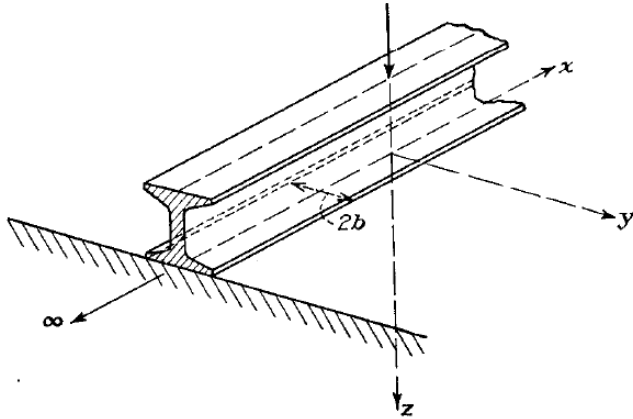
$$k_{fs} = k_h B = \frac{1.3E_s}{1-\nu_s^2} \left( \frac{E_s B^4}{E_p I_p} \right)^{1/12} \quad (3-8-90)$$

where

$E_p$ : Young's modulus of a pile,  $I_p$ : Moment of inertia of a pile

$E_s$ : Young's modulus of soil,  $\nu_p$ : Poisson ratio of soil

This formula is based on the study by Biot (1937) with respect to the ground spring against bending of an infinite beam on ground and is modified by Vesic (1961). Francis extended this concept to the pile considered that there is ground on both sides of the beam and doubled the ground stiffness.



#### Reference:

- 1) Francis A. J, Analysis of Pile Groups with Flexural Resistance, Journal of the Soil Mechanics and Foundations Division, 1964, Vol. 90, Issue 3, Pg. 1-32
- 2) Biot, M. A. Bending of an infinite beam on an elastic foundation. J. Appl. Mech., 1937, 4, 1, A1-A7
- 3) Vesic A.B, Bending of beams resting on isotropic elastic solid, Journal of the Engineering Mechanics Division, 1961, Vol. 87, Issue 2, Pg. 35-54

### b-2) Horizontal damping of a single pile

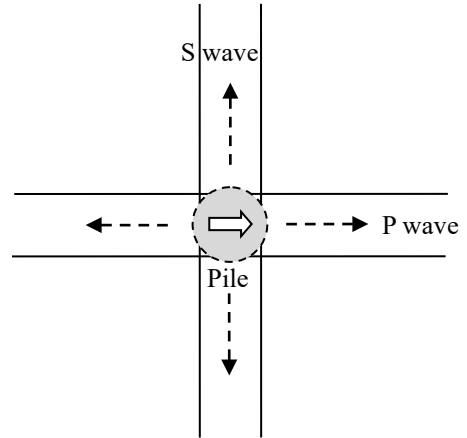
Gazetas proposed the following formula for the horizontal damping per unit length of a single pile:

$$c_{gs} = 2\rho_s B(V_L + V_s) \quad (3-8-91)$$

where

$$V_L = \frac{3.4V_s}{\pi(1-\nu_s)} : \text{Lysmer analog wave}$$

This damping expresses the radiation damping in both directions of the pile.



**Reference:** Gazetas, G. and Dobry, R, Horizontal Response of Piles in Layered Soils, J. Geo tech. Engrg. Div.,ASCE, Vol.110, pp.20-40, 1984

### b-3) Ground spring and damping coefficient between multiple layers

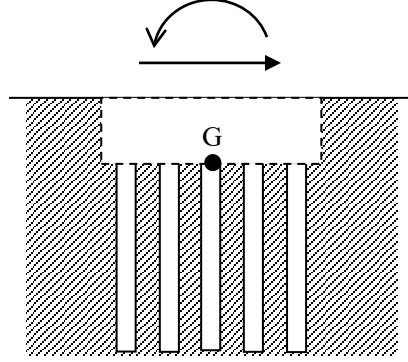
The ground spring and damping coefficient between multiple layers can be calculated by multiplying the layer thickness of each layer and averaging as

$$k'_{jsi} = 0.5(k_{js(i-1)}H_{i-1} + k_{jsi}H_i) \quad (3-8-92)$$

$$c'_{gsi} = 0.5(c_{gs(i-1)}H_{i-1} + c_{gsi}H_i) \quad (3-8-93)$$

### c) Impedance of group piles

In case of group piles, the impedance of the foundation can not be obtained from the simple addition of the impedances of individual piles because of the interaction of piles.



#### c-1) Group effect in horizontal direction (stiffness)

The horizontal stiffness of group piles is obtained from the horizontal stiffness of a single pile as,

$$K_{HG} = N_p \beta_H K_{HS} \quad (3-8-94)$$

where

$K_{HG}$  : horizontal stiffness of group piles,  $K_{HS}$  : horizontal stiffness of a single pile

$N_p$  : number of piles,  $\beta_H$  : coefficient of group effect

The following formula is adopted for STERA\_3D as the coefficient of group effect for horizontal ( $x$ ) direction,

$$\beta_{Hx} = 0.4 (S/B)^{0.3} (N_x/2)^{-0.74(S/B)^{-0.43}} (N_y/2)^{-0.59(S/B)^{-0.54}} \quad (3-8-95)$$

where

$S$  : distance between piles in x-direction,  $B$  : diameter of pile,

$N_x$ ,  $N_y$  : number of piles in x-direction and y-direction

The horizontal stiffness of a single pile is obtained from Eq. (3-8-89) as

$$K_{HS} = (4E_p I_p)^{1/4} \bar{k}_S^{3/4} \quad (3-8-96)$$

where

$\bar{k}_S$  : the stiffness coefficient of a single pile under homogenous ground

The horizontal stiffness of group piles

$$K_{HG} = N_p \beta_H K_{HS} = N_p \beta_H (4E_p I_p)^{1/4} \bar{k}_S^{3/4} = (4N_p E_p I_p)^{1/4} \bar{k}_G^{3/4} \quad (3-8-97)$$

$$\bar{k}_G = N_p \beta_H^{4/3} \bar{k}_S \quad (3-8-98)$$

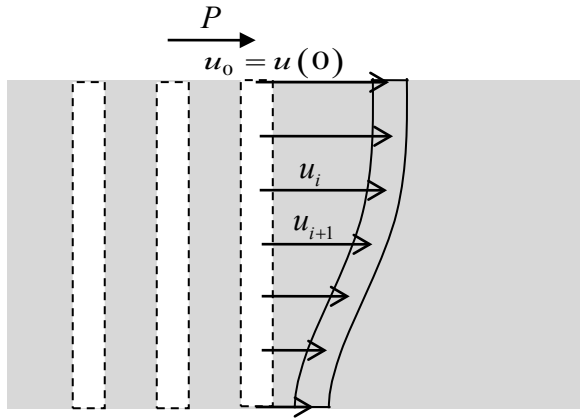
For the horizontal damping, the group effect is assumed negligible, and the horizontal damping of group piles is obtained as,

$$c_{HG} = N_p c_{HS} \quad (3-8-99)$$

where

$c_{HG}$  : damping coefficient of group piles,  $c_{HS}$  : damping coefficient of a single pile

In evaluating the horizontal ground stiffness of the group pile  $K_{HG}$  in layered ground, it is necessary to determine the value of the stiffness coefficient  $\bar{k}_G$  which represents the average stiffness coefficient in layered ground. The following iterative procedure is used to calculate  $\bar{k}_G$ .



Step. 1 Set the initial value of  $\bar{k}_G$  as

$$\bar{k}_G = \text{average of } k_{Gi} \text{ in the surface layer } (< 5B)$$

where  $k_{Gi} = N_p \beta_H^{4/3} k_{Si}$  : horizontal stiffness of group pile at i-th layer from Eq.(3-8-89)

Step. 2 The flexural deformation of a pile under the horizontal load  $P$  at the top is approximated by

$$u = \frac{P}{4N_p E_p I_p \beta^3} e^{-\beta x} (\sin \beta x + \cos \beta x), \quad \beta = \sqrt[4]{\frac{\bar{k}_G}{4EI}} \quad (3-8-100)$$

The horizontal stiffness at the top can be calculated by,

$$K_{HG2} = \frac{\sum k_{Gi} u_i}{u_0} \quad (3-8-101)$$

Step. 3 Update  $\bar{k}_G$  as

$$\bar{k}_G = (K_{HG2})^{4/3} / (4N_p E_p I_p)^{1/4} \quad (3-8-102)$$

Step. 4 Go back to Step 1 until  $K_{HG2} \approx K_{HG}$ .

### c-2) Group effect in horizontal direction (damping)

The damping effect of the soil material is considered as

$$k_{Gi}^* = k_{Gi} (1 + 2ih_{Gi}) \quad (3-8-103)$$

where  $h_{Gi}$  is the damping factor of the soil in i-th layer. The horizontal damping at the top of group piles can be calculated by,

$$h_{HG} = \frac{\sum h_{Gi} u_i}{u_0} \quad (3-8-104)$$

Therefore, the imaginary part of the horizontal stiffness is

$$K_{HG}' \approx 2h_{HG} K_{HG}$$

In the same way, the horizontal radiation damping at the top of group piles can be calculated by

$$c_{HG} = \frac{\sum c_{Gi} u_i}{u_0} \quad (3-8-105)$$

where  $c_{Gi} = N_p c_{Si}$

### c-3) Group effect in rocking direction (stiffness)

The group effect in rotational direction is assumed negligible and the coefficient of group effect is one. Therefore, the rotational stiffness is calculated from the vertical stiffness of individual pile as

$$K_{RGx} = \sum_{i=1}^m K_{Vi} y_i^2 : \text{around x-axis} \quad (3-8-106)$$

$$K_{RGy} = \sum_{i=1}^m K_{Vi} x_i^2 : \text{around y-axis} \quad (3-8-107)$$

where

$x_i, y_i$  : distance from the center of rotation in  $x, y$  directions

### c-4) Group effect in rocking direction (damping)

In case of rocking direction, the damping effect of the soil at the bottom of the pile is considered dominant.

$$h_{RG} = h_b \quad (3-8-108)$$

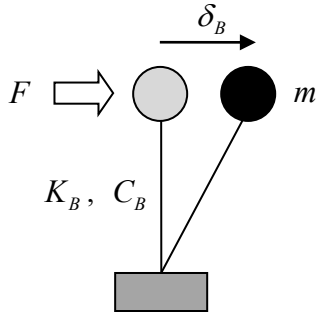
where  $h_b$  : damping factor of the soil at the bottom of the pile

Therefore, the imaginary part of the rocking stiffness is

$$K_{RG}' \approx 2h_{RG} K_{RG} \quad (3-8-109)$$

### 3.8.8 Equivalent period and damping factor considering soil structure interaction

#### a) Equivalent period



Force and deformation

$$\delta_B = F/K_B$$

$$\delta = \delta_B = F/K_B$$

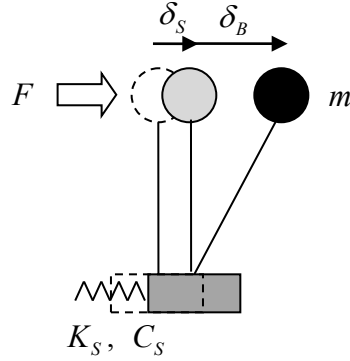
Stiffness

$$K = K_B$$

Period (mass of foundation is ignored)

$$T_B = 2\pi\sqrt{\frac{m}{K_B}}$$

$$T = 2\pi\sqrt{\frac{m}{K}} = T_B$$



$$\delta_S = F/K_S$$

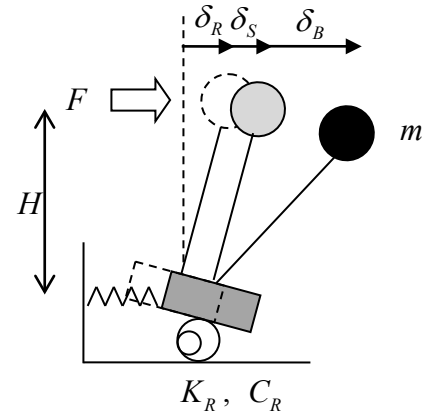
$$\delta = \delta_B + \delta_S = F(1/K_B + 1/K_S)$$

$$K = \frac{1}{1/K_B + 1/K_S}$$

$$T_S = 2\pi\sqrt{\frac{m}{K_S}}$$

$$T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{m}{K_B} + \frac{m}{K_S}}$$

$$= \sqrt{T_B^2 + T_S^2}$$



$$\delta_R = H\theta_R = H(M/K_R)$$

$$= H(FH/K_R) = F/(K_R/H^2)$$

$$\delta = \delta_B + \delta_S + \delta_R$$

$$= F(1/K_B + 1/K_S + 1/(K_R/H^2))$$

$$K = \frac{1}{1/K_B + 1/K_S + 1/(K_R/H^2)}$$

$$T_R = 2\pi\sqrt{\frac{mH^2}{K_R}}$$

$$T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{m}{K_B} + \frac{m}{K_S} + \frac{mH^2}{K_R}}$$

$$= \sqrt{T_B^2 + T_S^2 + T_R^2}$$

## b) Equivalent damping

### b-1) Equivalent damping for material damping

Force including damping force is

$$F = C\dot{\delta} + K\delta$$

For a harmonic excitation  $\delta = ae^{i\omega t}$

$$\dot{\delta} = i\omega\delta$$

Then

$$F = K\left(1 + i\omega\frac{C}{K}\right)\delta = K(1 + 2hi)\delta$$

where  $h$  is the damping ratio

$$h = \frac{\omega C}{2K}$$

Defining the viscous damping ratio separately for each dashpot,

$$h_B = \frac{\omega_B C_B}{2K_B}, \quad h_S = \frac{\omega_S C_S}{2K_S}, \quad h_R = \frac{\omega_R C_R}{2K_R}$$

This is the case to define the damping force to be **independent to the frequency of excitation**.

This type of damping is called “**material damping**”.

Total complex stiffness will be

$$\frac{1}{K(1 + 2hi)} = \frac{1}{K_B(1 + 2h_B i)} + \frac{1}{K_S(1 + 2h_S i)} + \frac{1}{K_R/H^2(1 + 2h_R i)}$$

Using the relationship

$$\frac{1}{1 + 2hi} = \frac{1 - 2hi}{(1 + 2hi)(1 - 2hi)} = \frac{1 - 2hi}{1 + 4h^2} \approx 1 - 2hi$$

Then

$$\frac{1}{K}(1 - 2hi) = \frac{1}{K_B}(1 - 2h_B i) + \frac{1}{K_S}(1 - 2h_S i) + \frac{1}{K_R/H^2}(1 - 2h_R i)$$

From the real part

$$\frac{1}{K} = \frac{1}{K_B} + \frac{1}{K_S} + \frac{1}{K_R/H^2}$$

From the imaginary part

$$h = \frac{K}{K_B}h_B + \frac{K}{K_S}h_S + \frac{K}{K_R/H^2}h_R = \left(\frac{T_B}{T}\right)^2 h_B + \left(\frac{T_S}{T}\right)^2 h_S + \left(\frac{T_R}{T}\right)^2 h_R$$

## b-2) Equivalent damping for viscous damping

Force including damping force is

$$F = C\dot{\delta} + K\delta = m\left(\frac{C}{m}\dot{\delta} + \frac{K}{m}\delta\right) = m(2h\omega\dot{\delta} + \omega^2\delta)$$

For a harmonic excitation  $\delta = ae^{ipt}$

$$F = m(\omega^2 + 2h\omega pi)\delta = m\omega^2\left(1 + 2h\left(\frac{p}{\omega}\right)i\right)\delta$$

This is the case to define the damping force to be **dependent to the frequency of excitation**.

This type of damping is called “**viscous damping**”.

Total complex stiffness will be

$$\frac{1}{m\omega^2\left(1 + 2h\left(\frac{p}{\omega}\right)i\right)} = \frac{1}{m\omega_B^2\left(1 + 2h_B\left(\frac{p}{\omega_B}\right)i\right)} + \frac{1}{m\omega_S^2\left(1 + 2h_S\left(\frac{p}{\omega_S}\right)i\right)} + \frac{1}{m\omega_R^2\left(1 + 2h_R\left(\frac{p}{\omega_R}\right)i\right)}$$

Using the relationship

$$\frac{1}{1 + 2h\left(\frac{p}{\omega}\right)i} \approx 1 - 2h\left(\frac{p}{\omega}\right)i$$

Then

$$\frac{1}{m\omega^2}\left(1 - 2h\left(\frac{p}{\omega}\right)i\right) = \frac{1}{m\omega_B^2}\left(1 - 2h_B\left(\frac{p}{\omega_B}\right)i\right) + \frac{1}{m\omega_S^2}\left(1 - 2h_S\left(\frac{p}{\omega_S}\right)i\right) + \frac{1}{m\omega_R^2}\left(1 - 2h_R\left(\frac{p}{\omega_R}\right)i\right)$$

From the real part

$$\frac{1}{\omega^2} = \frac{1}{\omega_B^2} + \frac{1}{\omega_S^2} + \frac{1}{\omega_R^2} \rightarrow \frac{1}{K} = \frac{1}{K_B} + \frac{1}{K_S} + \frac{1}{K_R/H^2}$$

From the imaginary part

$$h\left(\frac{p}{\omega}\right) = \frac{K}{K_B}h_B\left(\frac{p}{\omega_B}\right) + \frac{K}{K_S}h_S\left(\frac{p}{\omega_S}\right) + \frac{K}{K_R/H^2}h_R\left(\frac{p}{\omega_R}\right)$$

In case of the resonance frequency,  $p = \omega$

$$h = \left(\frac{\omega}{\omega_B}\right)^3 h_B + \left(\frac{\omega}{\omega_S}\right)^3 h_S + \left(\frac{\omega}{\omega_R}\right)^3 h_R$$

or

$$h = \left(\frac{T_B}{T}\right)^3 h_B + \left(\frac{T_S}{T}\right)^3 h_S + \left(\frac{T_R}{T}\right)^3 h_R$$

## 4. Freedom Vector

### 4.1 Node freedom

Each node has six degrees of freedom and the freedom number is defined as shown in the figure below.

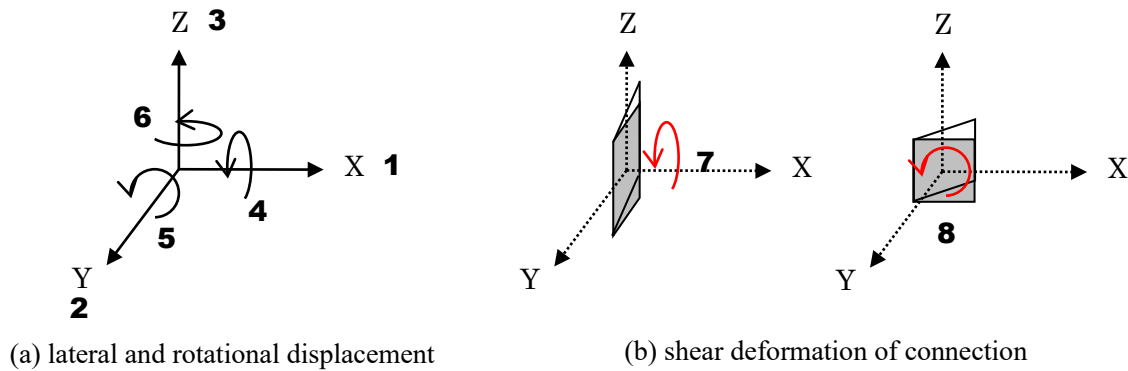


Figure 4-1-1 Global coordinate

### 4.2 Freedom vector

The freedom vector is defined to indicate the number of all freedoms of the structure, where the restrained freedom is set to be zero. For the structure in the figure below, the freedom vector has zero components for the fixed nodes (Nodes 1-4) and eight components for other nodes (Nodes 5-8). Therefore, the total number of freedom of the structure is  $8 \times 4 = 32$ .

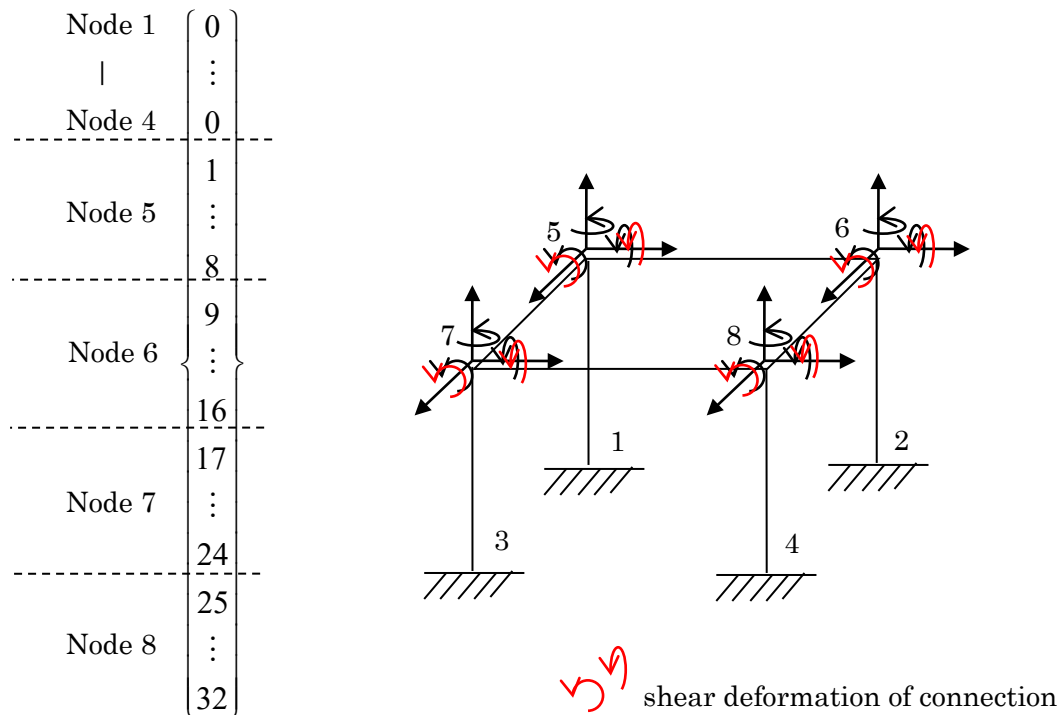


Figure 4-2-1 Example of the freedom vector

### 4.3 Dependent freedom

#### (1) Rigid floor assumption

In the default setting, the floor diaphragm is assumed to be rigid for the in-plane deformation. Therefore, the in-plane freedoms at the nodes in a floor are represented by the freedoms at the center of gravity of the same floor.

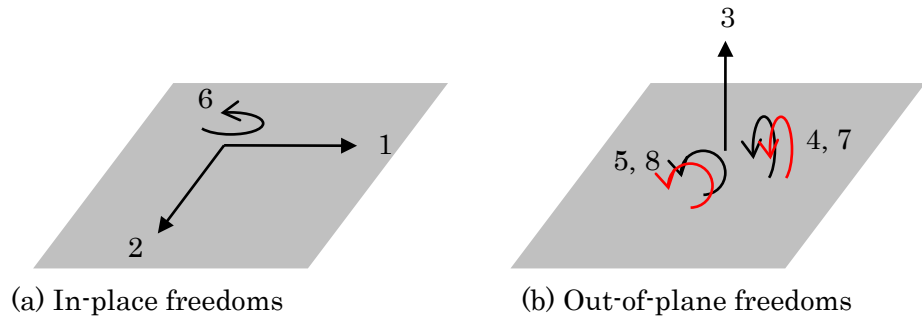


Figure 4-3-1 In-plane and out-of-plane freedom

For example, the in-plane freedoms at the node A in Figure 4-3-2 are expressed by the in-plane freedoms at the center of gravity G as follows:

$$\begin{Bmatrix} u_{xA} \\ u_{yA} \\ \theta_{zA} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & l_{yA} \\ 0 & 1 & -l_{xA} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{xG} \\ u_{yG} \\ \theta_{zG} \end{Bmatrix} \quad (4-3-1)$$

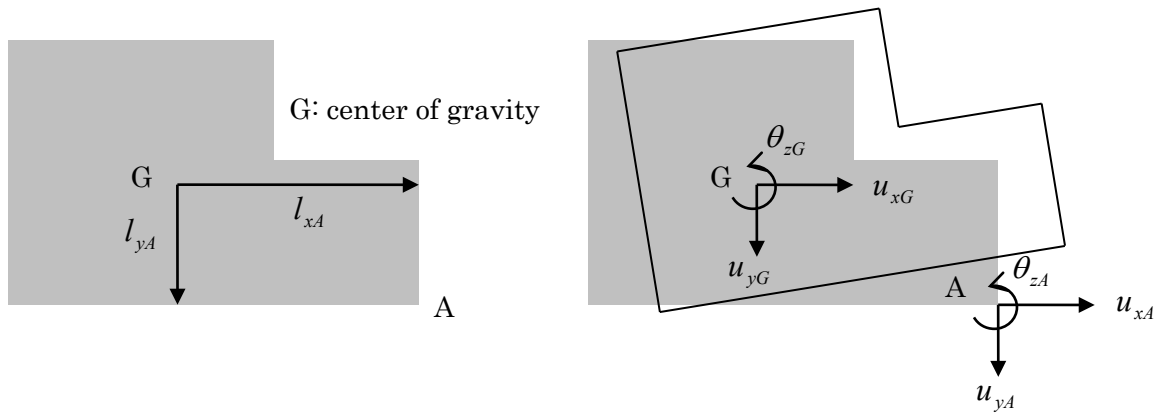


Figure 4-3-2 Rigid floor assumption

In case of the structure in the figure below, in addition to the original nodes, new nodes for the center of gravity are defined as “Node 5” and “Node 10”. Under the rigid floor assumption, the freedom vector has zero components for the in-plane freedoms at the nodes except the center of gravity. Therefore, the total number of independent freedom is 23.

Node 1-5	$\begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix}$
Node 6	$\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ 4 \\ 5 \end{Bmatrix}$
Node 7	$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 7 \\ 8 \\ 0 \\ 9 \\ 10 \end{Bmatrix}$
Node 8	$\begin{Bmatrix} 0 \\ 0 \\ 11 \\ 12 \\ 13 \\ 0 \\ 14 \\ 15 \end{Bmatrix}$
Node 9	$\begin{Bmatrix} 0 \\ 0 \\ 16 \\ 17 \\ 18 \\ 0 \\ 19 \\ 20 \end{Bmatrix}$
Node 10	$\begin{Bmatrix} 21 \\ 22 \\ 0 \\ 0 \\ 0 \\ 23 \\ 0 \\ 0 \end{Bmatrix}$

Freedom vector

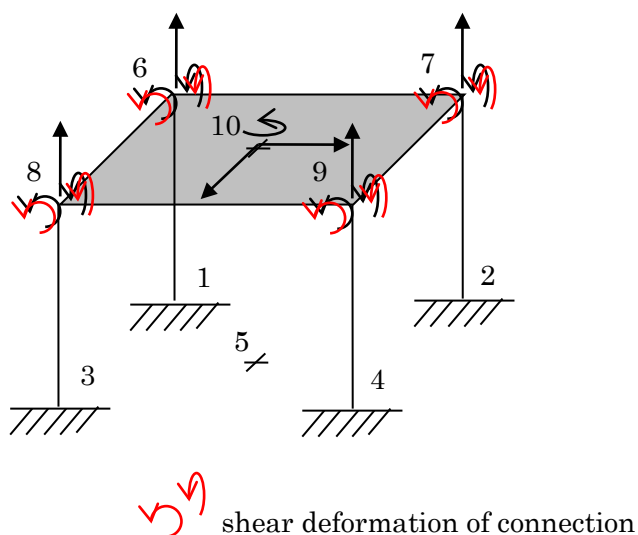


Figure 4-3-3 Example of the freedom vector with rigid floor assumption

(2) Including wall element

The wall element model has rigid beams at the top and bottom of the wall, therefore, as shown in Figure 4-3-4, the rotation angles in the wall panel plane,  $\theta_{y1}$  and  $\theta_{y2}$ , are dependent to the vertical displacements,  $\delta_{z1}$  and  $\delta_{z2}$ . Also, the horizontal displacement in the wall panel plane,  $u_{x2}$ , is dependent to the displacement,  $u_{x1}$ . The connection is assumed to be rigid.

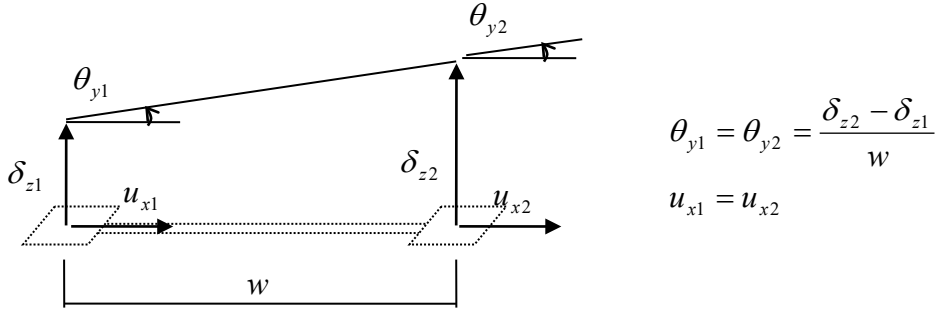


Figure 4-3-4 Relationship between node displacements for a wall element (X-wall)

In a matrix form;

$$\begin{Bmatrix} u_{x1} \\ \theta_{y1} \\ \theta_{y2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/w & 1/w \\ 0 & -1/w & 1/w \end{bmatrix} \begin{Bmatrix} u_{x2} \\ \delta_{z1} \\ \delta_{z2} \end{Bmatrix} \quad (4-3-2)$$

In case of Y-direction wall, the relationship can be written as;

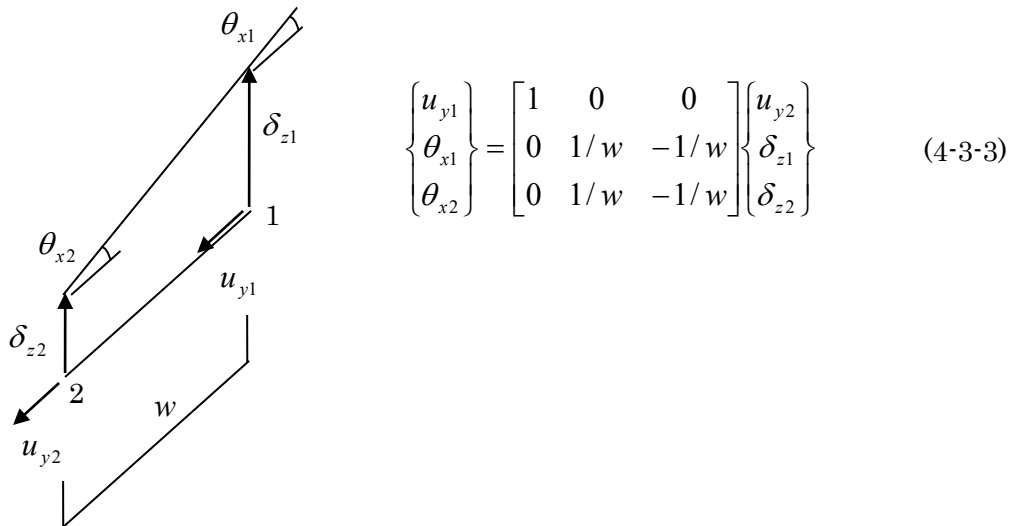


Figure 4-3-5 Relationship between node displacements for a wall element (Y-wall)

For example, in case of the structure in the figure below, by eliminating dependent freedoms, the total number of freedom becomes 17.

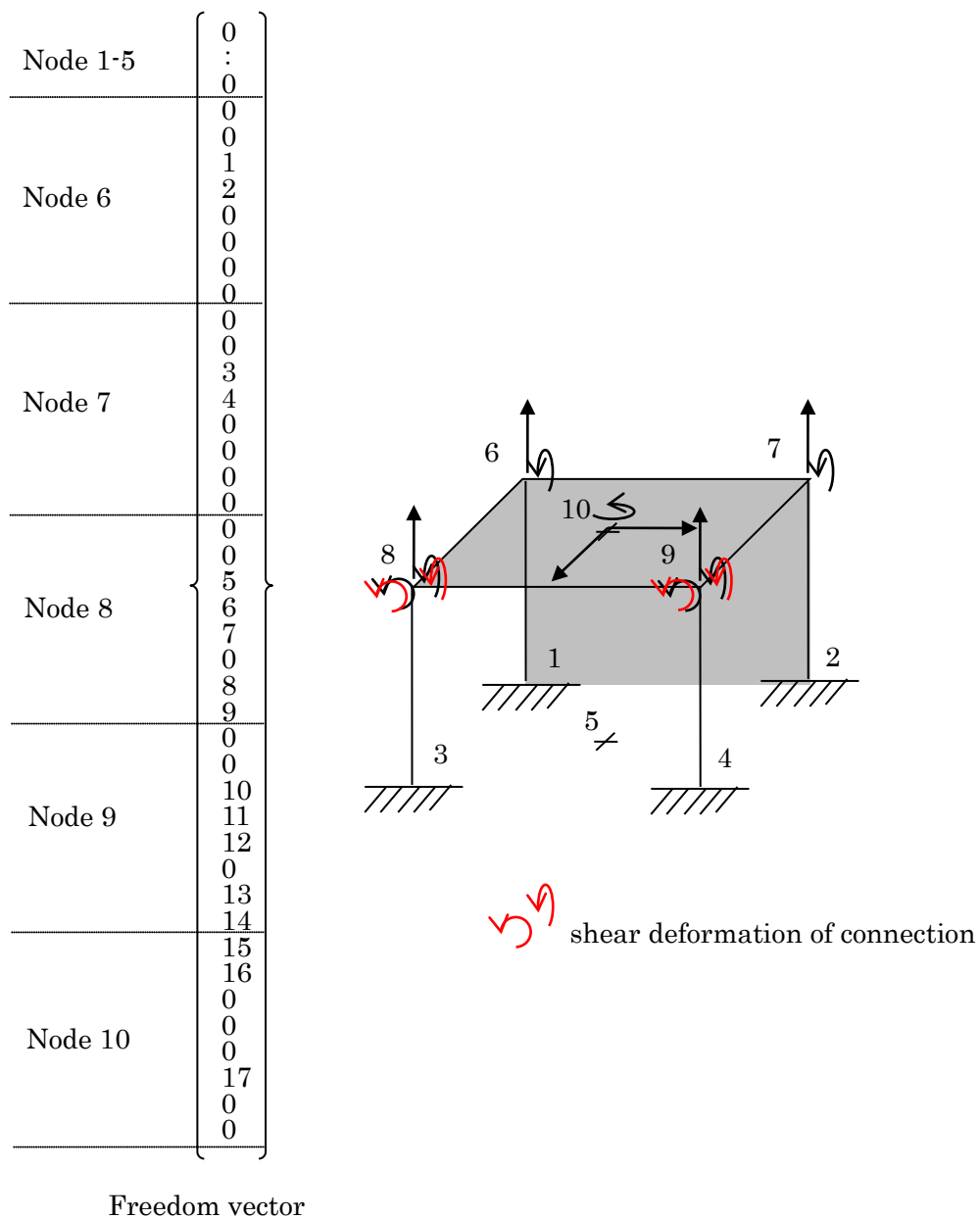
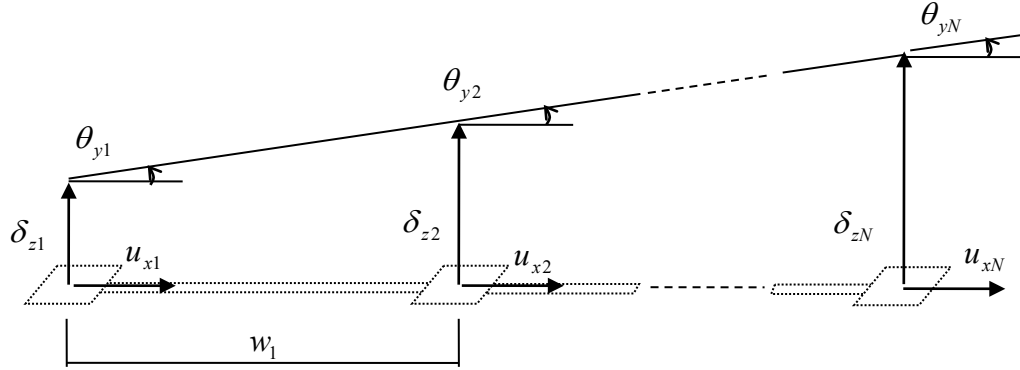


Figure 4-3-6 Example of the freedom vector with a wall element

### (3) Series of walls

In case several walls are joined together in series, it is assumed that all walls are connected by rigid beams at the top and bottom. Therefore, as shown in Figure 4-3-7, the rotation angles in the wall panel plane,  $\theta_{y1}$  and  $\theta_{y2}$ , are dependent to the vertical displacements,  $\delta_{z1}$  and  $\delta_{z2}$ . Also, the horizontal displacement in the wall panel plane,  $u_{x2}$ , is dependent to the displacement,  $u_{x1}$ . The connection is assumed to be rigid.



$$\theta_{y1} = \theta_{y2} = \dots = \theta_{yN} = \frac{\delta_{zN} - \delta_{z1}}{L}, \quad L = \sum_{k=1}^{N-1} w_k$$

$$\delta_{zi} = \delta_{z1} + \theta_{yi} L_i = (1 - L_i / L) \delta_{z1} + (L_i / L) \delta_{zN}, \quad L_i = \sum_{k=1}^{i-1} w_k$$

$$u_{x1} = u_{x2} = \dots = u_{xN}$$

Figure 4-3-7 Series of wall connected by a rigid beam (X-wall)

In a matrix form;

$$\theta_{yi} = \begin{bmatrix} -1/L & 1/L \end{bmatrix} \begin{Bmatrix} \delta_{z1} \\ \delta_{zN} \end{Bmatrix} \quad (4-3-4)$$

$$\delta_{zi} = \begin{bmatrix} 1 - L_i / L & L_i / L \end{bmatrix} \begin{Bmatrix} \delta_{z1} \\ \delta_{zN} \end{Bmatrix} \quad (4-3-5)$$

In case of Y-direction wall, the relationship can be written as;

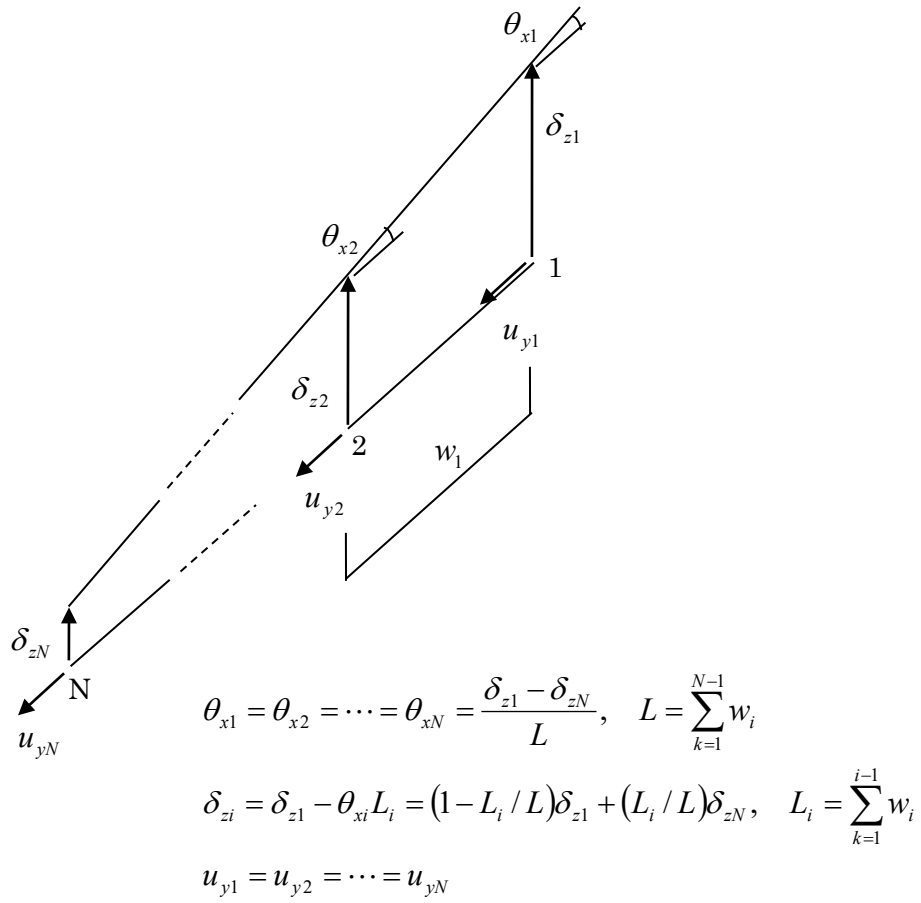


Figure 4-3-8 Series of wall connected by a rigid beam (Y-wall)

In a matrix form;

$$\theta_{xi} = \begin{bmatrix} 1/L & -1/L \end{bmatrix} \begin{Bmatrix} \delta_{z1} \\ \delta_{zN} \end{Bmatrix} \quad (4-3-6)$$

$$\delta_{zi} = \begin{bmatrix} 1 - L_i / L & L_i / L \end{bmatrix} \begin{Bmatrix} \delta_{z1} \\ \delta_{zN} \end{Bmatrix} \quad (4-3-7)$$

#### (4) Including ground springs

In case there are ground springs (sway and rocking springs) at the basement of the building, the floor diaphragm of the basement is assumed to be rigid for both in-plane and out-of-plane deformation and the freedoms other than sway and rocking freedoms are restricted at the center of gravity.

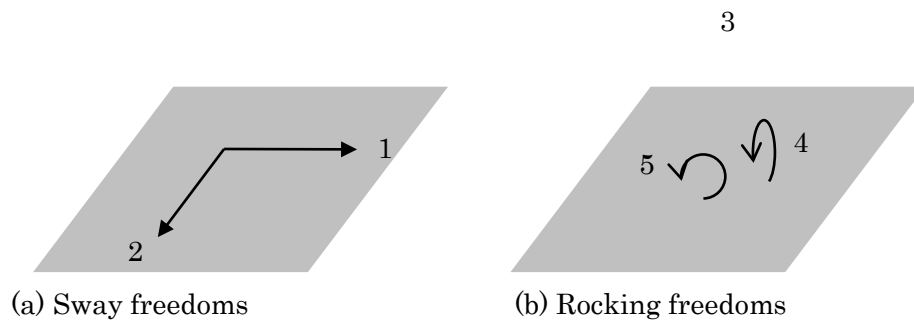


Figure 4-3-9 Freedoms of ground springs

In case of the structure in the Figure below, by eliminating dependent freedoms, the total number of freedom becomes 21.

Node 1-4	0 ⋮ 0
Node 5	1 2 0 3 4 0 0 0
Node 6	0 0 5 6 0 0 0 0
Node 7	0 0 7 8 0 0 0 0
Node 8	0 0 9 10 11 0 12 13
Node 9	0 0 14 15 16 0 17 18
Node 10	19 20 0 0 0 21 0 0

Freedom vector

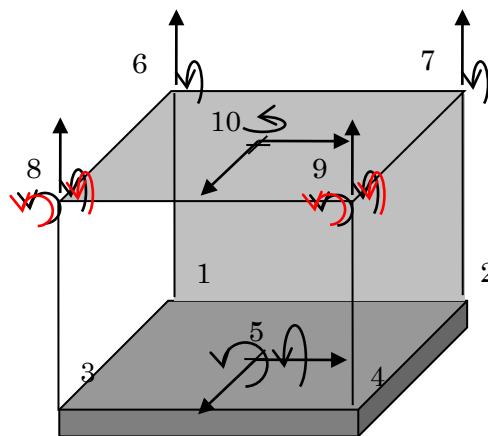


Figure 4-3-10 Example of the freedom vector with ground springs



In the same way, for the case of including wall elements, Equation (4-3-2) expresses the relationship between dependent freedom and independent freedom, that is;

$$\underbrace{\begin{Bmatrix} u_{x1} \\ \theta_{y1} \\ \theta_{y2} \end{Bmatrix}}_{\text{Dependent freedom}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/w & 1/w \\ 0 & -1/w & 1/w \end{bmatrix} \underbrace{\begin{Bmatrix} u_{x2} \\ \delta_{y1} \\ \delta_{y2} \end{Bmatrix}}_{\text{Independent freedom}}$$

It can be arranged into the transformation matrix between the freedom vectors of all nodes;

$$\underbrace{j \begin{Bmatrix} \vdots \\ \vdots \\ \vdots \\ u_{x1} \\ \vdots \\ \vdots \\ \theta_{y1} \\ \vdots \\ \vdots \\ \theta_{y2} \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix}}_{\text{Dependent freedom}} = \underbrace{\begin{bmatrix} & & & p & & & q & & & r \\ & & & 0 & & & 1 & & & 0 \\ & & & -1/w & & & 0 & & & 1/w \\ & & & -1/w & & & 0 & & & 1/w \end{bmatrix}}_{[T_I]} \underbrace{\begin{Bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \delta_{y1} \\ \vdots \\ \vdots \\ u_{x2} \\ \vdots \\ \vdots \\ \delta_{y2} \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix}}_{\text{Independent freedom}} \begin{matrix} p \\ q \\ r \end{matrix}$$

The components of two matrices,  $[N_I]$  and  $[F_I]$  will be;

$$[N_I] = j \begin{bmatrix} p & r & 0 \end{bmatrix} \quad ; \text{ Matrix for independent freedom numbers}$$

$$[F_I] = j \begin{bmatrix} -1/w & -1/w & 0 \end{bmatrix} \quad ; \text{ Matrix for transformation components from independent freedoms}$$

Initial conditions of  $[N_I]$  and  $[F_I]$  are:

$$[N_I] = i \begin{bmatrix} i & 0 & 0 \end{bmatrix}, \quad [F_I] = i \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

In Figure 4-4-1 (rigid connection), the element node displacement vector of the beam element between Node 8 and Node 9 is,

$$\{u_{z8} \quad u_{z9} \quad \theta_{y8} \quad \theta_{y9} \quad \delta_{x8} \quad \delta_{x9}\}^T \quad (4-4-1)$$

Those displacements correspond to the location numbers in the freedom vector as;

$$\{u_{z8} \quad u_{z9} \quad \theta_{y8} \quad \theta_{y9} \quad \delta_{x8} \quad \delta_{x9}\}^T \Rightarrow \{45 \quad 51 \quad 47 \quad 53 \quad 43 \quad 49\}^T \quad (4-4-2)$$

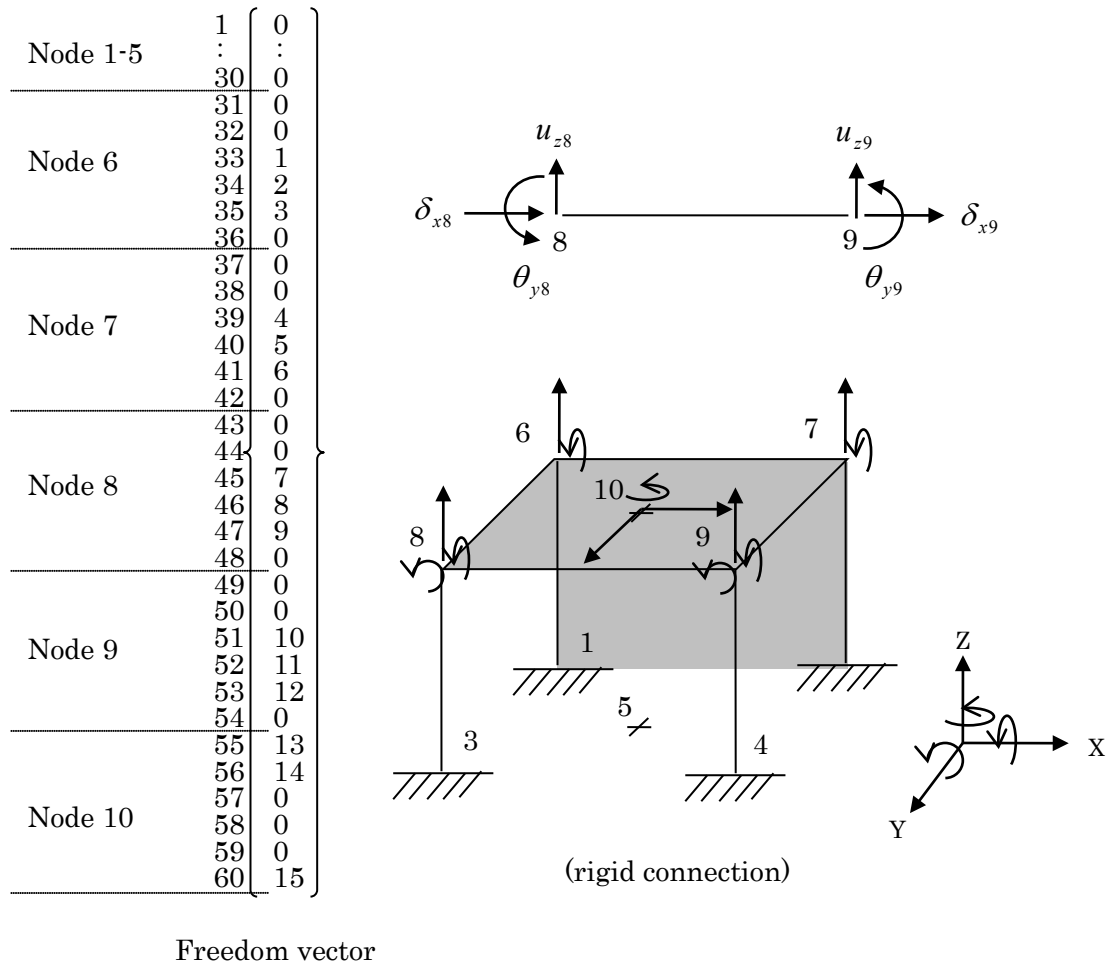


Figure 4-4-1 Example of location matrix for beam element

From rigid floor assumption, the components of independent matrices,  $[N_I]$  and  $[F_I]$  will be;

$$[N_I] = \begin{Bmatrix} \vdots \\ 43 \\ 45 \\ 47 \\ 49 \\ 51 \\ 53 \\ \vdots \end{Bmatrix} \begin{bmatrix} 55 & 60 & 0 \\ 45 & 0 & 0 \\ 47 & 0 & 0 \\ 55 & 60 & 0 \\ 51 & 0 & 0 \\ 53 & 0 & 0 \end{bmatrix}, \quad [F_I] = \begin{Bmatrix} \vdots \\ 43 \\ 45 \\ 47 \\ 49 \\ 51 \\ 53 \\ \vdots \end{Bmatrix} \begin{bmatrix} 1 & l_{y8} & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & l_{y9} & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (4-4-3)$$

From the matrix,  $[N_I]$ , the freedoms of (43) and (49) are replaced to the independent freedoms (55) and (60). Therefore, the independent location numbers and freedom numbers of the beam element are:

$$\begin{aligned} & \{u_{z8} \quad u_{z9} \quad \theta_{y8} \quad \theta_{y9} \quad \delta_{x8} \quad \delta_{x9}\}^T \\ & \Rightarrow \{45 \quad 51 \quad 47 \quad 53 \quad 43 \quad 49\}^T \\ & \Rightarrow \{45 \quad 51 \quad 47 \quad 53 \quad 55 \quad 60\}^T; \text{ independent location number} \\ & \Rightarrow \{u_{z8} \quad u_{z9} \quad \theta_{y8} \quad \theta_{y9} \quad u_{x10} \quad \theta_{z10}\}^T \\ & \Rightarrow \{5 \quad 8 \quad 7 \quad 10 \quad 11 \quad 13\}^T; \text{ freedom number} \end{aligned} \quad (4-4-4)$$

The transformation from independent displacements (= global node displacements) to element node displacements is obtained from the matrix,  $[F_I]$ , as follows:

$$\begin{Bmatrix} u_{z8} \\ u_{z9} \\ \theta_{y8} \\ \theta_{y9} \\ \delta_{x8} \\ \delta_{x9} \end{Bmatrix} = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 & l_{y8} \\ 0 & & & & 1 & l_{y9} \end{bmatrix} \begin{Bmatrix} u_{z8} \\ u_{z9} \\ \theta_{y8} \\ \theta_{y9} \\ u_{x10} \\ \theta_{z10} \end{Bmatrix} = [T_{ixB}] \begin{Bmatrix} u_{z8} \\ u_{z9} \\ \theta_{y8} \\ \theta_{y9} \\ u_{x10} \\ \theta_{z10} \end{Bmatrix} \quad (4-4-5)$$

#### 4.5 Stiffness matrix corresponding to independent degrees of freedom

The constitutive equation of the beam element and formulation of global stiffness matrix from element stiffness matrix are shown below:

$$\begin{Bmatrix} P_{z8} \\ P_{z9} \\ M_{y8} \\ M_{y9} \\ P_{x10} \\ M_{z10} \end{Bmatrix} = \begin{matrix} 5 & 8 & 7 & 10 & 11 & 13 \\ \begin{bmatrix} k_{5,5} & k_{5,8} & k_{5,7} & k_{5,10} & k_{5,11} & k_{5,13} \\ & k_{8,8} & k_{8,7} & k_{8,10} & k_{8,11} & k_{8,13} \\ & & k_{7,7} & k_{7,10} & k_{7,11} & k_{7,13} \\ & & & k_{10,10} & k_{10,11} & k_{10,13} \\ & sym. & & & k_{11,11} & k_{11,13} \\ & & & & & k_{12,12} \end{bmatrix} \end{matrix} \begin{Bmatrix} u_{z8} \\ u_{z9} \\ \theta_{y8} \\ \theta_{y9} \\ u_{x10} \\ \theta_{z10} \end{Bmatrix}$$

Element stiffness matrix

Locate element stiffness according to the freedom number

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{bmatrix} & \begin{bmatrix} & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & k_{5,5} & & k_{5,7} & k_{5,8} & k_{5,10} & k_{5,11} & & k_{5,13} \\ & & & & & & & & & & & & \\ & & & & & & k_{7,7} & k_{7,8} & k_{7,10} & k_{7,11} & & k_{7,13} \\ & & & & & & & k_{8,8} & k_{8,10} & k_{8,11} & & k_{8,13} \\ & & & & & & & & & & & & \\ & & & & & & & & & k_{10,10} & k_{10,11} & & k_{10,13} \\ & & & & & & & & & & k_{11,11} & & k_{11,13} \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & k_{13,13} \end{bmatrix} \end{matrix}$$

Global stiffness matrix

Figure 4-5-1 Formulation of global stiffness matrix

In general, the transformation from independent displacements (= global node displacements) to element node displacements for the X-beam is described as Equation (2-1-13).

$$\begin{Bmatrix} u_{zA} \\ u_{zB} \\ \theta_{yA} \\ \theta_{yB} \\ \delta_{xA} \\ \delta_{xB} \end{Bmatrix} = [T_{ixB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-1-13)$$

And the constitutive equation of the X-beam is also described in Equation (2-1-20).

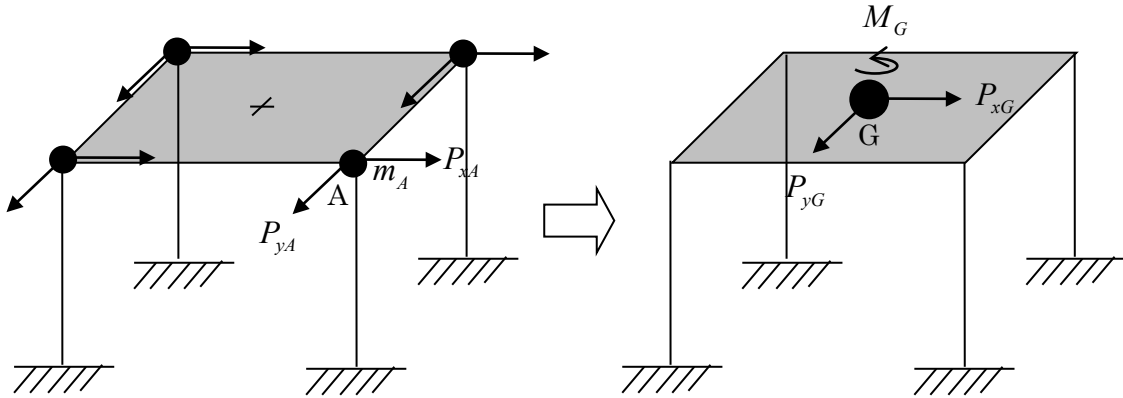
$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix} = [K_{xB}] \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad (2-1-20)$$

Using the same procedure in Figure 4-5-1, the element stiffness matrix is added into the global stiffness matrix.

#### 4.6 Mass matrix corresponding to independent degrees of freedom

Mass is assigned in each node. The inertia force at the node will be also transformed according to the transformation of the variables. Here, the rotational inertia at each node is ignored.

##### (1) Rigid floor assumption

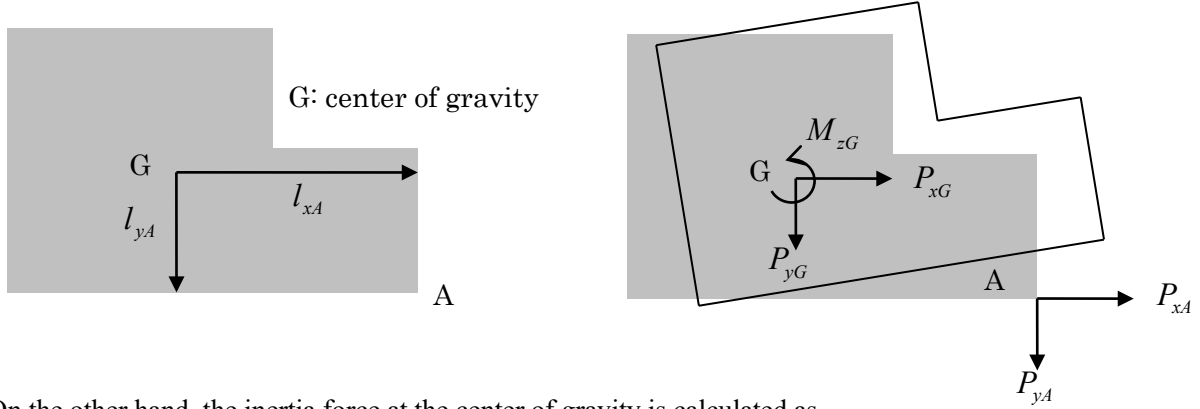


The inertia force at the node A is

$$\begin{Bmatrix} P_{xA} \\ P_{yA} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -m_A \ddot{u}_{xA} \\ -m_A \ddot{u}_{yA} \\ 0 \end{Bmatrix} = - \begin{bmatrix} m_A & 0 & 0 \\ 0 & m_A & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{xA} \\ \ddot{u}_{yA} \\ \ddot{\theta}_{zA} \end{Bmatrix} \quad (4-5-1)$$

Under the rigid floor assumption, the in-plane freedoms at the nodes in a floor are represented by the freedoms at the center of gravity of the same floor. Therefore,

$$\begin{Bmatrix} \ddot{u}_{xA} \\ \ddot{u}_{yA} \\ \ddot{\theta}_{zA} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & l_{yA} \\ 0 & 1 & -l_{xA} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{xG} \\ \ddot{u}_{yG} \\ \ddot{\theta}_{zG} \end{Bmatrix} = [T_A] \begin{Bmatrix} \ddot{u}_{xG} \\ \ddot{u}_{yG} \\ \ddot{\theta}_{zG} \end{Bmatrix}, \quad [T_A] = \begin{bmatrix} 1 & 0 & l_{yA} \\ 0 & 1 & -l_{xA} \\ 0 & 0 & 1 \end{bmatrix} \quad (4-5-2)$$



On the other hand, the inertia force at the center of gravity is calculated as,

$$\begin{Bmatrix} P_{xG} \\ P_{yG} \\ M_{zG} \end{Bmatrix} = \begin{Bmatrix} P_{xA} \\ P_{yG} \\ l_{yA}P_{xA} - l_{xA}P_{yG} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{yA} & -l_{xA} & 1 \end{bmatrix} \begin{Bmatrix} P_{xA} \\ P_{yG} \\ 0 \end{Bmatrix} = [T_A]^T \begin{Bmatrix} P_{xA} \\ P_{yG} \\ 0 \end{Bmatrix} \quad (4-5-3)$$

Therefore,

$$\begin{aligned} \begin{Bmatrix} P_{xG} \\ P_{yG} \\ M_{zG} \end{Bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{yA} & -l_{xA} & 1 \end{bmatrix} \begin{Bmatrix} P_{xA} \\ P_{yA} \\ 0 \end{Bmatrix} = -[T_A]^T \begin{bmatrix} m_A & 0 & 0 \\ 0 & m_A & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{xA} \\ \ddot{u}_{yA} \\ \ddot{\theta}_{zA} \end{Bmatrix} \\ &= -[T_A]^T \begin{bmatrix} m_A & 0 & 0 \\ 0 & m_A & 0 \\ 0 & 0 & 0 \end{bmatrix} [T_A] \begin{Bmatrix} \ddot{u}_{xG} \\ \ddot{u}_{yG} \\ \ddot{\theta}_{zG} \end{Bmatrix} = - \begin{bmatrix} m_A & 0 & l_{yA}m_A \\ 0 & m_A & -l_{xA}m_A \\ l_{yA}m_A & -l_{xA}m_A & (l_{xA}^2 + l_{yA}^2)m_A \end{bmatrix} \begin{Bmatrix} \ddot{u}_{xG} \\ \ddot{u}_{yG} \\ \ddot{\theta}_{zG} \end{Bmatrix} \end{aligned} \quad (4-5-4)$$

If we ignore the off-diagonal components,

$$\begin{Bmatrix} P_{xG} \\ P_{yG} \\ M_{zG} \end{Bmatrix} = - \begin{bmatrix} m_A & 0 & 0 \\ 0 & m_A & 0 \\ 0 & 0 & m_A(l_{xA}^2 + l_{yA}^2) \end{bmatrix} \begin{Bmatrix} \ddot{u}_{xG} \\ \ddot{u}_{yG} \\ \ddot{\theta}_{zG} \end{Bmatrix} \quad (4-5-5)$$


Taking the sum of the inertia force from the all nodes at the same floor,

$$\begin{Bmatrix} P_{xG} \\ P_{yG} \\ M_{zG} \end{Bmatrix} = - \begin{bmatrix} m_G & 0 & 0 \\ 0 & m_G & 0 \\ 0 & 0 & I_G \end{bmatrix} \begin{Bmatrix} \ddot{u}_{xG} \\ \ddot{u}_{yG} \\ \ddot{\theta}_{zG} \end{Bmatrix}, \quad m_G = \sum_i^N m_i, \quad I_G = \sum_i^N m_i (l_{ix}^2 + l_{iy}^2) \quad (4-5-6)$$

where, N is the total number of the nodes at the floor.

(2) Including rigid beam

The wall element model has rigid beams at the top and bottom of the wall, and the horizontal displacement in the wall panel plane,  $u_{x2}$ , is dependent to the displacement,  $u_{x1}$ .



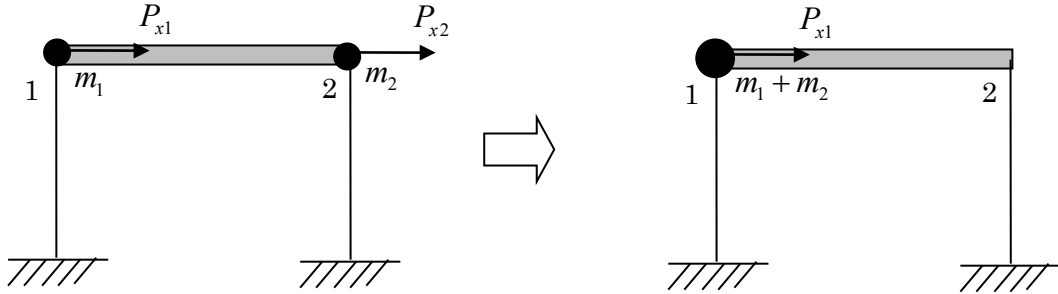
$$\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = [T] \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}$$

The inertia force after transformation is

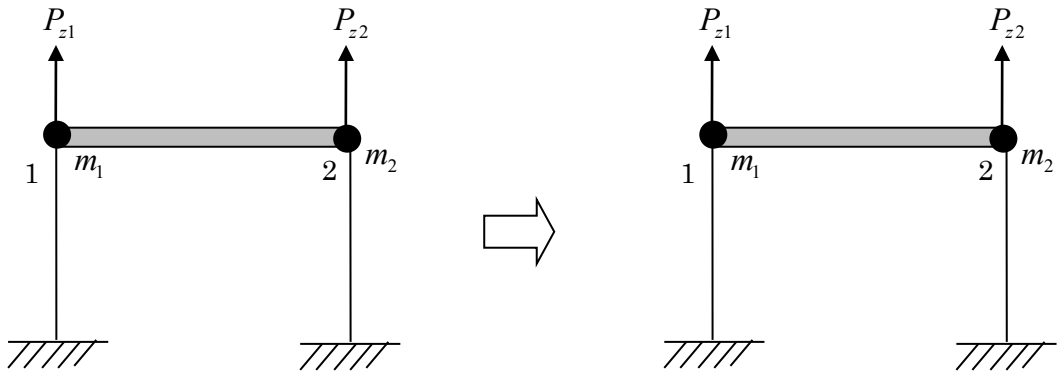
$$\begin{Bmatrix} P_{x1} \\ P_{x2} \end{Bmatrix} = [T]^T \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} [T] \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix} = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}$$

Therefore, the horizontal mass is

$$P_{x1} = (m_1 + m_2) \ddot{u}_{x1}$$

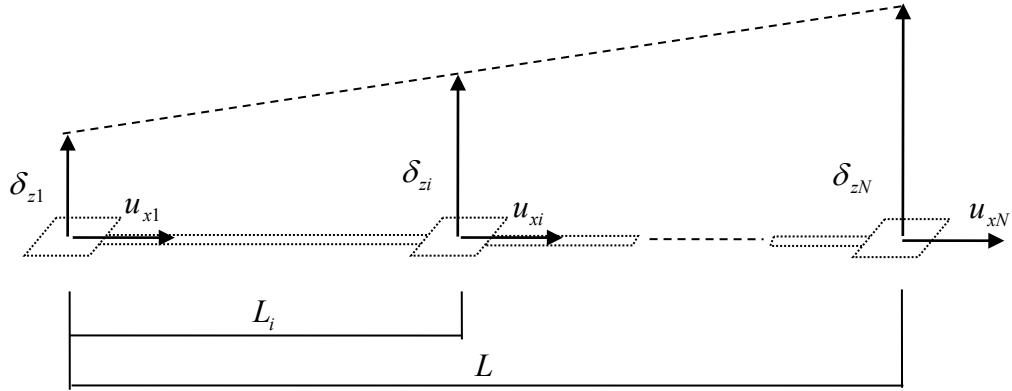


On the other hand, the vertical mass is the same as before.



### (3) Series of rigid beams

In case several walls are joined together in series, it is assumed that all walls are connected by rigid beams at the top and bottom.



The all horizontal displacements at the nodes are dependent to the horizontal displacement of the first node,  $u_{x1}$ .

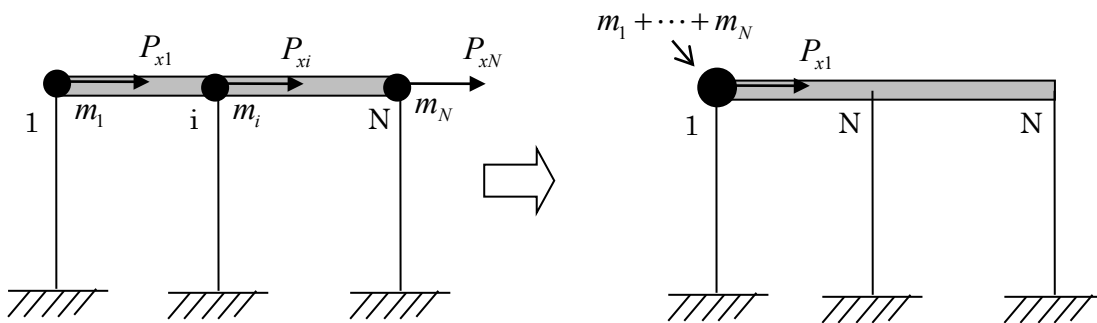
$$u_{x1} = u_{x2} = \dots = u_{xN}$$

Also, the vertical displacement at the middle node  $\delta_{zi}$  is dependent to the vertical displacements of the nodes at both ends,  $\delta_{z1}$   $\delta_{zN}$ .

$$\delta_{zi} = \left(1 - \frac{L_i}{L}\right) \delta_{z1} + \left(\frac{L_i}{L}\right) \delta_{zN}$$

Therefore, the horizontal mass is

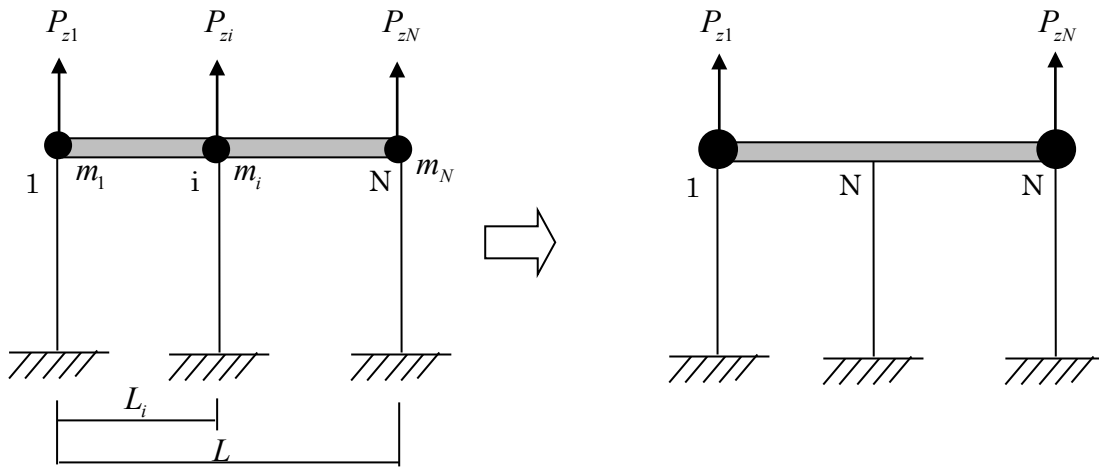
$$P_{x1} = (m_1 + m_2 + \dots + m_N) \ddot{u}_{x1} = \left(\sum_{i=1}^N m_i\right) \ddot{u}_{x1}$$



The vertical mass is

$$P_{z1} = \left( \sum_{i=1}^N \left( 1 - \frac{L_i}{L} \right) m_i \right) \ddot{\delta}_{z1}$$

$$P_{zN} = \left( \sum_{i=1}^N \left( \frac{L_i}{L} \right) m_i \right) \ddot{\delta}_{zN}$$



## 5. Equation of motion

### 5.1 Mass matrix

In the default setting, the mass at each node is identical and equally distributed as

$$M_i = \frac{1}{N_{floor}} M_{floor} \quad (5-1-1)$$

where,  $M_i$ : mass at the node  $i$ ,  $M_{floor}$ : total mass of the floor,  $N_{floor}$ : total number of nodes in the floor.

However, you can change the mass at each node depending on the place of the node by setting “proportion to influence area” in Option Menu. In this case, the mass at each node is determined from the following equation:

$$M_i = \frac{A_i}{A_{floor}} M_{floor} \quad (5-1-1)$$

where,  $A_i$ : influence area of node  $i$ ,  $A_{floor}$ : total area of the floor. Influence area of the node is different depending on the place of the node as shown in Figure 5-1-1.

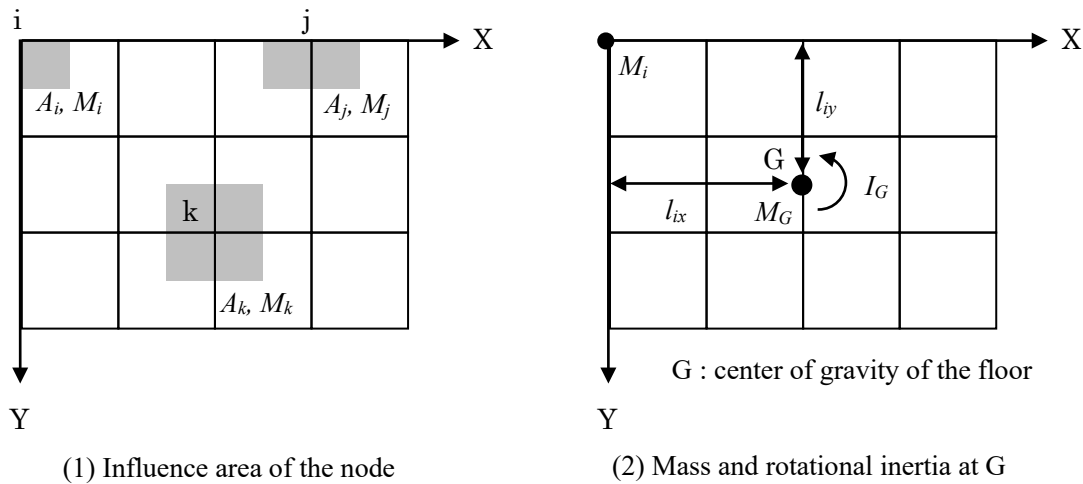


Figure 5-1-1 Mass and rotational inertia at the node

The process to determine the mass based on influence area is as follows:

Step 1. Calculate the slab area (block with cross mark)

Step 2. The area of the block is divided equally to the corner nodes. (Figure 5-1-2)

Step 3. If there is no corner node, the area is divided equally to the all nodes in a floor. (Figure 5-1-3)

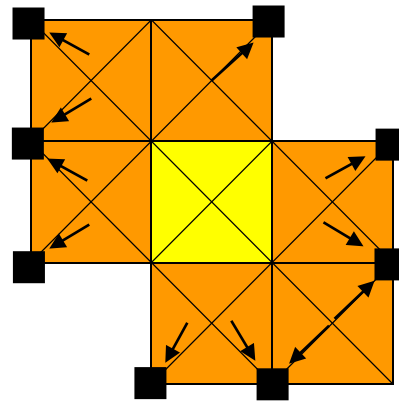
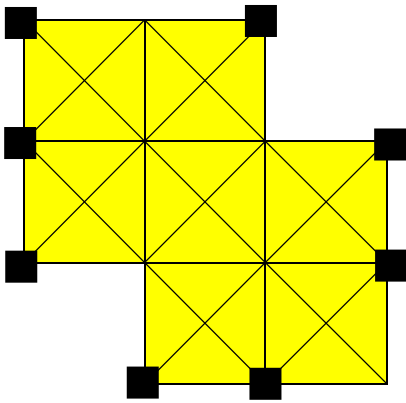


Figure 5-1-2. Influence area of the node (red)

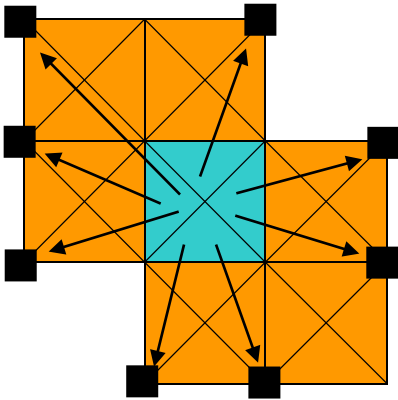
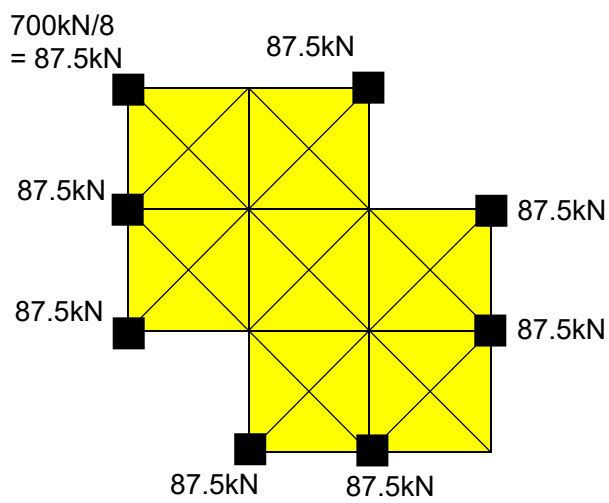
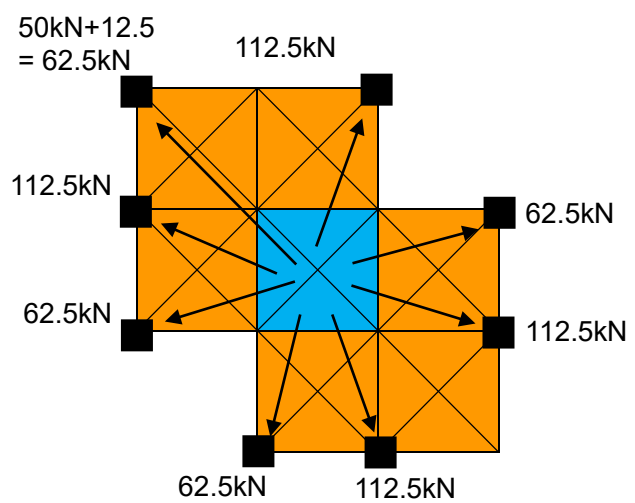


Figure 5-1-3. Distribution of the rest area

Example) Floor weight = 700kN



(a) Same for all nodes



(b) Proportional to influence area

Figure 5-1-4 Example of mass distribution

$$M_G = M_{floor} \tag{5-1-2}$$

The rotational inertia at the center of gravity,  $I_G$ , along the z-axis is obtained from the following equation:

$$I_G = \sum_i^N M_i (l_{ix}^2 + l_{iy}^2) \quad (5-1-3)$$

$$I_i = 0, \quad i = 1, \dots, N \quad (5-1-4)$$

The mass matrix is obtained as,

$$[M] = \begin{bmatrix} \ddots & & 0 & & & & & & & & 0 \\ & 0 & \ddots & & & & & & & & \\ & & & M_i & & & & & & & \\ & & & & M_i & & & & & & \\ & & & & & M_i & & & & & \\ & & & & & & I_i & & & & \\ & & & & & & & I_i & & & \\ & & & & & & & & I_i & & \\ & & & & & & & & & \ddots & 0 \\ & & & & & & & & & 0 & \ddots \\ 0 & & & & & & & & & & \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ \vdots \\ M_i \\ M_i \\ M_i \\ I_i \\ I_i \\ I_i \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \quad (5-1-5)$$

For example, the mass vector of the structure in Figure 5-1-5 will be as follows:

Node 6	$\left\{ \begin{array}{c} M_6 \\ 0 \end{array} \right\}$
Node 7	$\left\{ \begin{array}{c} M_7 \\ 0 \end{array} \right\}$
Node 8	$\left\{ \begin{array}{c} M_8 \\ 0 \\ 0 \end{array} \right\}$
Node 9	$\left\{ \begin{array}{c} M_9 \\ 0 \\ 0 \end{array} \right\}$
Node 10	$\left\{ \begin{array}{c} M_{10} \\ M_{10} \\ I_{10} \end{array} \right\}$

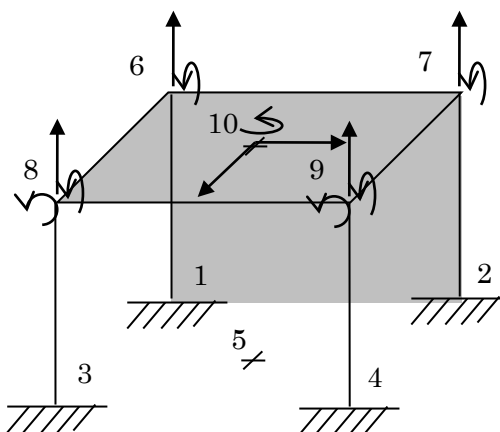
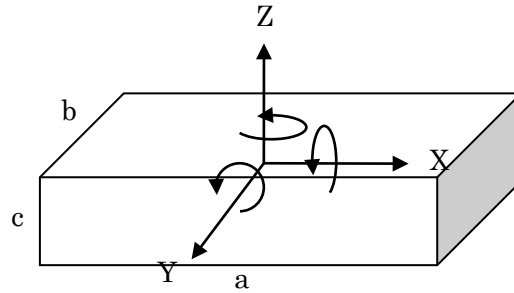
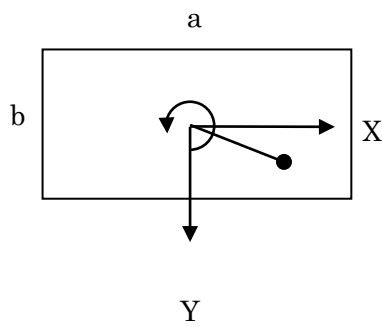


Figure 5-1-5 Example of mass vector

In case a complete rigid floor such as a foundation slab for the ground springs, we need to calculate the rotational inertia at the center of gravity along each axis.



The rotational inertia along Z-axis is



$$I_Z = \int \rho r^2 dV = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} \left( \frac{M}{abc} \right) (x^2 + y^2) dx dy dz$$

$$= \frac{M}{abc} \left( \int_{-a/2}^{a/2} x^2 dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz + \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} y^2 dy \int_{-c/2}^{c/2} dz \right)$$

$$= \frac{M}{12} (a^2 + b^2)$$

(5-1-6)

In the same way, the rotational inertia along X-axis is

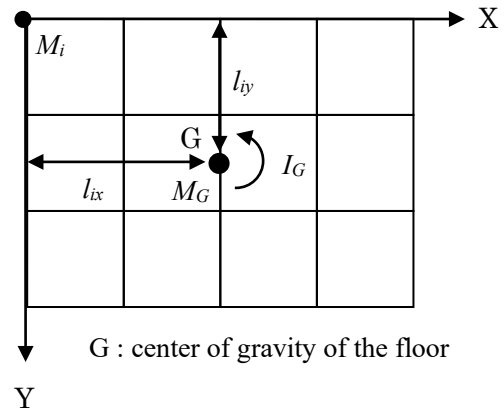
$$I_X = \frac{M}{12} (b^2 + c^2) \quad (5-1-7)$$

The rotational inertia along Y-axis is

$$I_Y = \frac{M}{12} (a^2 + c^2) \quad (5-1-8)$$

If the mass is located at each node, as already mentioned, the rotational inertia at the center of gravity,  $I_G$ , along the Z-axis is obtained as

$$I_Z = \int \rho r^2 dV = \sum M_i (l_{ix}^2 + l_{iy}^2) \quad (5-1-9)$$



## 5.2 Stiffness matrix

### 1) Global stiffness matrix

As shown in Figure 4-5-1, the global stiffness matrix  $[K]$  is formulated from the element stiffness matrices.

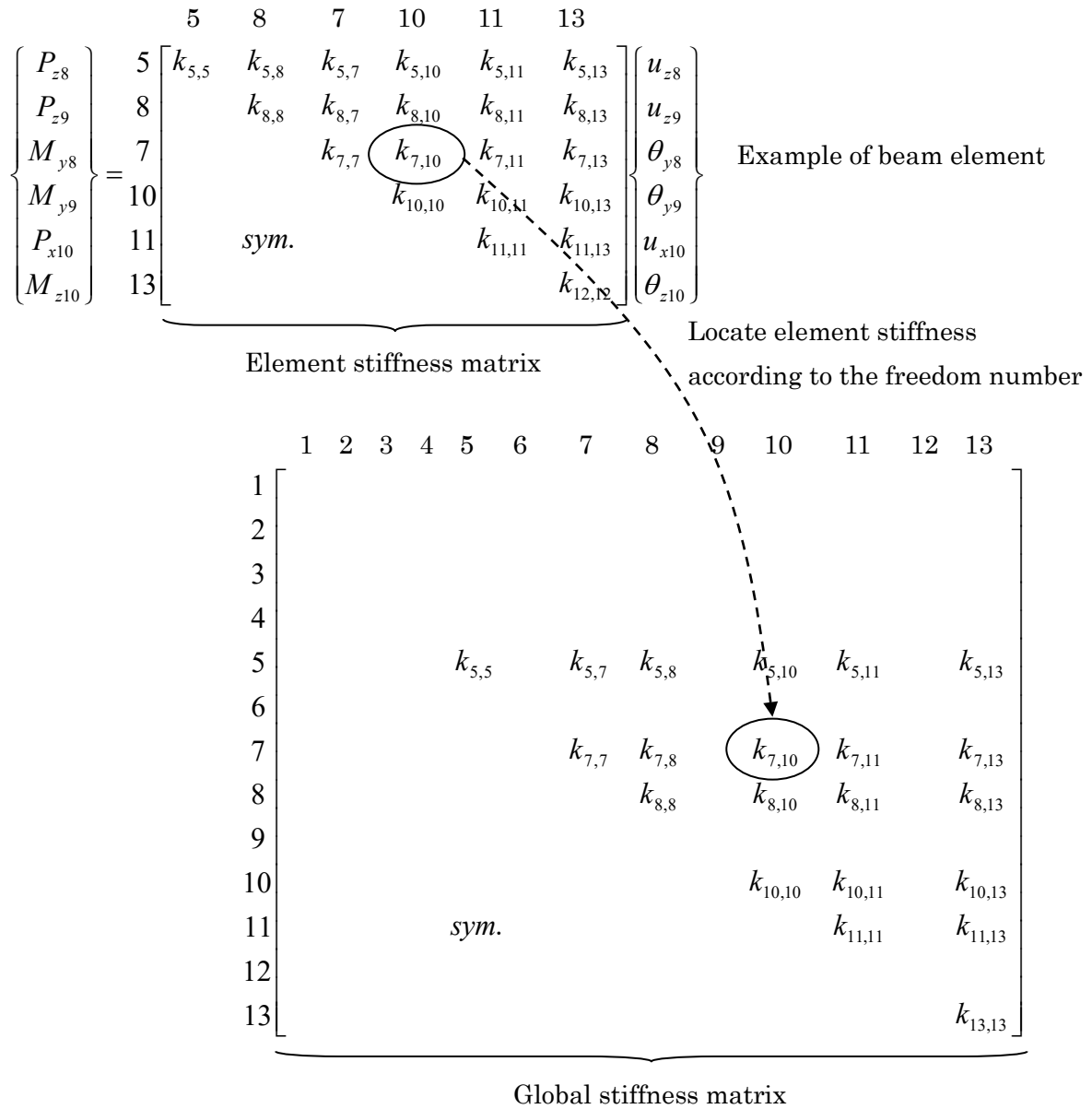
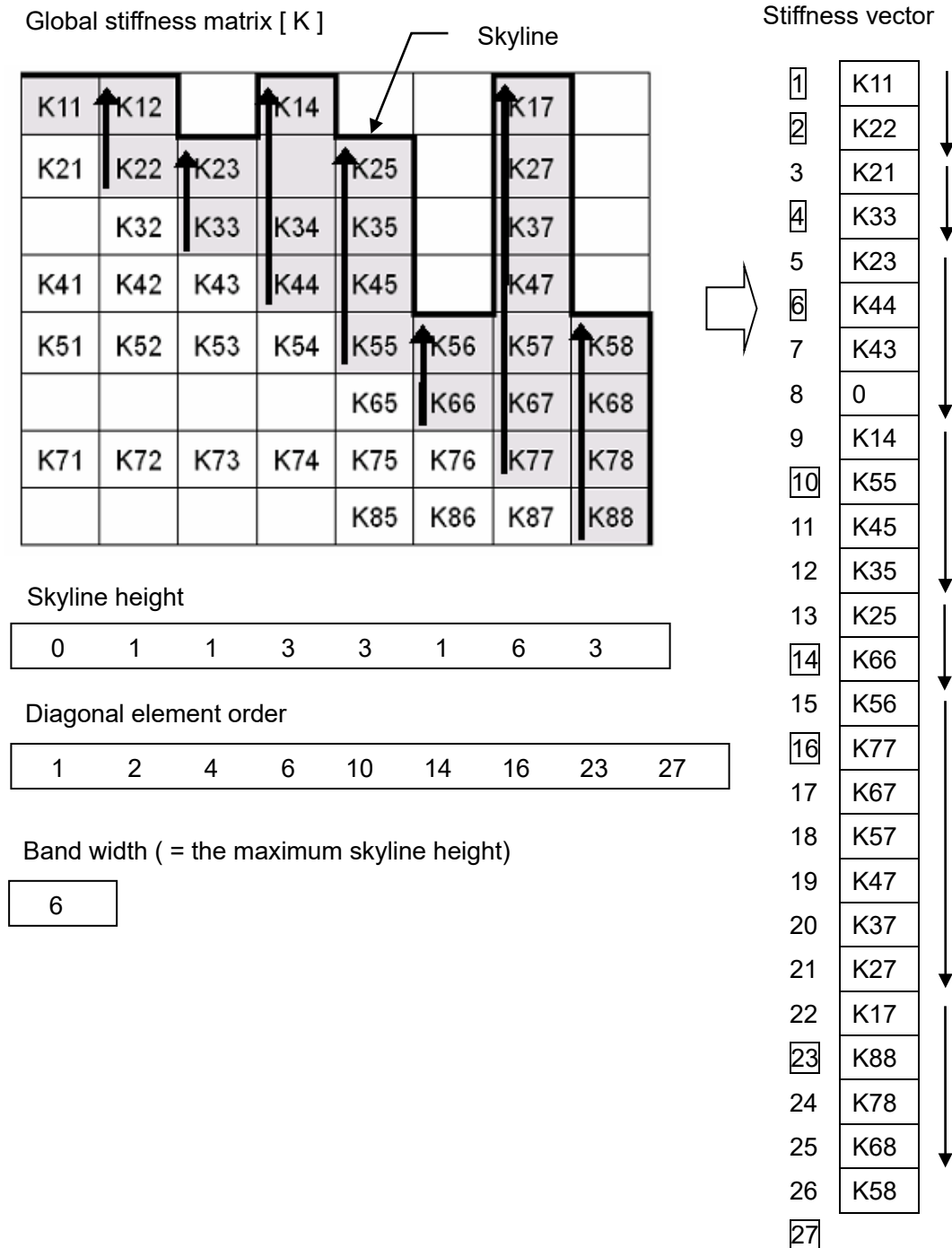


Figure 5-2-1 Formulation of global stiffness matrix

## 2) Skyline method

Usually, the global stiffness matrix  $[K]$  is symmetric and sparse as shown below. To save the memory size of the computer and to reduce the calculation time for the linear equation solver, the elements in the upper triangular part of the matrix under the Skyline (bold line) are stored in a one-dimension vector.



### 5.3 Modal analysis

#### 1) Eigen value problem

The free vibration equilibrium equation without damping is

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\} \quad (5-3-1)$$

where  $[K]$  is the stiffness matrix and  $[M]$  is the mass matrix in the form;

$$[M] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} \quad (5-3-2)$$

The solution can be postulated to be in the form

$$\{u\} = \{\phi\} e^{i\omega t} \quad (5-3-3)$$

where  $\{\phi\}$  is a vector of order  $n$ ,  $\omega$  is a frequency of vibration of the vector  $\{\phi\}$ .

Substituting into the equilibrium equation, the generalized eigen problem is obtained as,

$$[K]\{\phi\} = \omega^2 [M]\{\phi\} \quad (5-3-4)$$

This eigen problem yields the  $n$  eigen solutions  $(\omega_1^2, \{\phi_1\}), (\omega_2^2, \{\phi_2\}), \dots, (\omega_n^2, \{\phi_n\})$  where the eigen vectors are M-orthonormalized as,

$$\{\phi_i\}^T [M] \{\phi_j\} = \sum_{k=1}^n m_k \phi_{i,k} \phi_{j,k} = 0 \quad ; \quad i \neq j \quad (5-3-5)$$

$\therefore$ )

Let's assume two different set of eigen solutions  $(\omega_i^2, \{\phi_i\}), (\omega_j^2, \{\phi_j\})$ .

Form Equation (5-3-4),

$$\{\phi_i\}^T [K] \{\phi_j\} = \{\phi_i\}^T ([K] \{\phi_j\}) = \omega_j^2 \{\phi_i\}^T [M] \{\phi_j\} \quad (5-3-6)$$

Since  $[K]$  and  $[M]$  are the symmetric matrices,

$$\{\phi_i\}^T [K] \{\phi_j\} = \{\phi_j\}^T ([K] \{\phi_i\}) = \omega_i^2 \{\phi_j\}^T [M] \{\phi_i\} = \omega_j^2 \{\phi_i\}^T [M] \{\phi_j\} \quad (5-3-7)$$

Subtracting Equation (5-3-7) from Equation (5-3-6),

$$(\omega_i^2 - \omega_j^2) \{\phi_i\}^T [M] \{\phi_j\} = 0 \quad (5-3-8)$$

Since  $\omega_i \neq \omega_j$ , we obtain Equation (5-3-5).

The vector  $\{\phi_i\}$  is called the  $i$ -th mode shape vector, and  $\omega_i$  is the corresponding frequency of vibration.

## 2) Modal decomposition of equilibrium equation

Defining a matrix  $[\Phi]$  whose columns are the eigenvectors and a diagonal matrix  $[\Omega^2]$  which stores the eigenvalues on its diagonal as,

$$[\Phi] = [\{\phi_1\} \quad \{\phi_2\} \quad \cdots \quad \{\phi_n\}], \quad [\Omega^2] = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix} \quad (5-3-9)$$

We introduce the following transformation on the displacement vector of the equilibrium equation (5-5-2):

$$\{u(t)\} = [\Phi] \{q(t)\} \quad (5-3-10)$$

Then,

$$[M][\Phi]\{\ddot{q}\} + [C][\Phi]\{\dot{q}\} + [K][\Phi]\{q\} = \{P\} \quad (5-3-11)$$

Multiplying  $[\Phi]^T$ ,

$$[\Phi]^T [M][\Phi]\{\ddot{q}\} + [\Phi]^T [C][\Phi]\{\dot{q}\} + [\Phi]^T [K][\Phi]\{q\} = [\Phi]^T \{P\} \quad (5-3-12)$$

where

$$[\Phi]^T [M][\Phi] = [\bar{M}] = \begin{bmatrix} \bar{m}_1 & & & \\ & \bar{m}_2 & & \\ & & \ddots & \\ & & & \bar{m}_n \end{bmatrix}, \quad \bar{m}_i = \{\phi_i\}^T [M] \{\phi_i\} \quad (5-3-13)$$

$$[\Phi]^T [K][\Phi] = [\Omega][\Phi]^T [M][\Phi] = [\Omega][\bar{M}] = \begin{bmatrix} \bar{k}_1 & & & \\ & \bar{k}_2 & & \\ & & \ddots & \\ & & & \bar{k}_n \end{bmatrix} = [\bar{K}], \quad \bar{k}_i = \omega_i^2 \bar{m}_i \quad (5-3-14)$$

A damping matrix that is diagonalized by  $[\Phi]$  is called a classical damping matrix.

$$[\Phi]^T [C][\Phi] = [\bar{C}] = \begin{bmatrix} \bar{c}_1 & & & \\ & \bar{c}_2 & & \\ & & \ddots & \\ & & & \bar{c}_n \end{bmatrix} \quad (5-3-15)$$

where,  $[\bar{M}]$ ,  $[\bar{C}]$  and  $[\bar{K}]$  are called **generalized modal mass**, **modal damping** and **modal stiffness matrix**, respectively.

Therefore,

$$[\bar{M}]\{\ddot{q}\} + [\bar{C}]\{\dot{q}\} + [\bar{K}]\{q\} = [\Phi]^T \{P\} \quad (5-3-16)$$

It can be reduced to n- equations of the form

$$\bar{m}_i \ddot{q}_i(t) + \bar{c}_i \dot{q}_i(t) + \bar{k}_i q_i(t) = r_i(t) \quad (5-3-17)$$

$$\text{where } r_i(t) = \{\phi_i\}^T \{P(t)\} = -\{\phi_i\}^T [M][U] \begin{Bmatrix} \ddot{X}_0(t) \\ \ddot{Y}_0(t) \\ \ddot{Z}_0(t) \end{Bmatrix} \quad (5-3-18)$$

By setting  $\bar{c}_i / \bar{m}_i = 2h_i \omega_i$  and  $\bar{k}_i / \bar{m}_i = \omega_i^2$

$$\ddot{q}_i(t) + 2h_i \omega_i \dot{q}_i(t) + \omega_i^2 q_i(t) = -\{\beta_i\}^T \begin{Bmatrix} \ddot{X}_0(t) \\ \ddot{Y}_0(t) \\ \ddot{Z}_0(t) \end{Bmatrix} = -\{\beta_{i,x} \ddot{X}_0(t) + \beta_{i,y} \ddot{Y}_0(t) + \beta_{i,z} \ddot{Z}_0(t)\} \quad (5-3-19)$$

where

$$\{\beta_i\}^T = \frac{\{\phi_i\}^T [M][U]}{\{\phi_i\}^T [M]\{\phi_i\}} = \frac{\{\phi_i\}^T [M] \begin{bmatrix} \{U_x\} \\ \{U_y\} \\ \{U_z\} \end{bmatrix}}{\{\phi_i\}^T [M]\{\phi_i\}} = \{\beta_{i,x} \ \beta_{i,y} \ \beta_{i,z}\}^T \quad (5-3-20)$$

$$\beta_{i,x,y,z} = \frac{\{\phi_i\}^T [M]\{U_{x,y,z}\}}{\{\phi_i\}^T [M]\{\phi_i\}} \quad (5-3-21)$$

$\beta_{i,x,y,z}$  is called “**participation factor**” of i-th mode.

$\beta_{i,x,y,z}$  is the coefficient when you decompose the vector  $\{U_{x,y,z}\}$  into mode vectors as,

$$\{U_{x,y,z}\} = [\Phi]\{\beta_{x,y,z}\} = \sum_{i=1}^n \beta_{i,x,y,z} \{\phi_i\} \quad (5-3-22)$$

$\therefore$ )

Multiplying  $[\Phi]^T [M]$ ,

$$[\Phi]^T [M]\{U_{x,y,z}\} = [\Phi]^T [M][\Phi]\{\beta_{x,y,z}\} = [\bar{M}]\{\beta_{x,y,z}\} \quad (5-3-23)$$

Therefore,

$$\{\beta_{x,y,z}\} = [\bar{M}]^{-1} [\Phi]^T [M]\{U_{x,y,z}\} \quad (5-3-24)$$

It is equivalent to Equation (5-3-21).

Equation (5-3-17) can be decomposed again as,

$$\begin{aligned}\ddot{x}_i(t) + 2h_i\omega_i\dot{x}_i(t) + \omega_i^2x_i(t) &= -\ddot{X}_0(t) \\ \ddot{y}_i(t) + 2h_i\omega_i\dot{y}_i(t) + \omega_i^2y_i(t) &= -\ddot{Y}_0(t) \\ \ddot{z}_i(t) + 2h_i\omega_i\dot{z}_i(t) + \omega_i^2z_i(t) &= -\ddot{Z}_0(t)\end{aligned}\quad (5-3-25)$$

and

$$q_i(t) = \beta_{i,x}x_i(t) + \beta_{i,y}y_i(t) + \beta_{i,z}z_i(t) \quad (5-3-26)$$

Therefore, the displacement vector is obtained by superposing displacement responses of single-degree-of-freedom (SDOF) systems in each mode and each direction as,

$$\{u(t)\} = [\Phi]\{q(t)\} = \sum_{i=1}^n \{\phi_i\} q_i(t) = \sum_{i=1}^n \beta_{i,x} \{\phi_i\} x_i(t) + \sum_{i=1}^n \beta_{i,y} \{\phi_i\} y_i(t) + \sum_{i=1}^n \beta_{i,z} \{\phi_i\} z_i(t) \quad (5-3-27)$$

$\beta_{i,x} \{\phi_i\}$  is called “**participation vector**” of i-th mode in x-direction.

### 3) Effective modal mass

The kinematic energy of the vibration is calculated as,

$$E_i = \frac{1}{2} \{\dot{u}(t)\}^T [M] \{\dot{u}(t)\} \quad (5-3-28)$$

For simplicity, only the x-directional response is considered. Then, mode decomposition of the velocity vector is

$$\{\dot{u}(t)\} = [\Phi]\{\dot{q}(t)\} = \sum_{i=1}^n \beta_{i,x} \{\phi_i\} \dot{x}_i(t) \quad (5-3-29)$$

Substituting into equation (5-3-28), we obtain

$$\begin{aligned}E &= \frac{1}{2} \{\dot{u}(t)\}^T [M] \{\dot{u}(t)\} = \frac{1}{2} \{\dot{q}(t)\}^T [\Phi]^T [M] [\Phi] \{\dot{q}(t)\} = \frac{1}{2} \{\dot{q}(t)\}^T [\bar{M}] \{\dot{q}(t)\} \\ &= \frac{1}{2} (\beta_1^2 \bar{m}_1) \dot{x}_1^2 + \frac{1}{2} (\beta_2^2 \bar{m}_2) \dot{x}_2^2 + \cdots + \frac{1}{2} (\beta_n^2 \bar{m}_n) \dot{x}_n^2 \\ &= \frac{1}{2} m_{e,1} \dot{x}_1^2 + \frac{1}{2} m_{e,2} \dot{x}_2^2 + \cdots + \frac{1}{2} m_{e,n} \dot{x}_n^2\end{aligned}\quad (5-3-30)$$

where  $m_{e,i} = \beta_i^2 \bar{m}_i$  is called the **effective mass**.

That is, the kinetic energy of a structure can be decomposed into the sum of the kinetic energies of one-degree-of-freedom systems with effective masses in each mode.

$$E = e_{e,1} + e_{e,2} + \cdots + e_{e,n}, \quad e_{e,i} = \frac{1}{2} m_{e,i} \dot{x}_i^2 \quad (5-3-31)$$

Therefore, when determining which vibration mode is dominant, the ratio of effective mass to total

effective mass

$$\eta_i = m_{e,i} / \sum_{k=1}^n m_{e,k} \quad (5-3-32)$$

is sometimes used. This is called the **effective mass ratio**.

In addition, the following relationship holds.

$$m_{e,i} = \beta_i^2 \bar{m}_i = \beta_i \frac{\{\phi_i\}^T [M] \{U\}}{\{\phi_i\}^T [M] \{\phi_i\}} \bar{m}_i = \beta_i \{\phi_i\}^T [M] \{U\} = \frac{\left(\{\phi_i\}^T [M] \{U\}\right)^2}{\{\phi_i\}^T [M] \{\phi_i\}} \quad (5-3-33)$$

$$\sum_{i=1}^n m_{e,i} = \sum_{i=1}^n \beta_i \{\phi_i\}^T [M] \{U\} = \{U\}^T [M] \{U\} = \sum_{i=1}^n m_i \quad (5-3-34)$$

#### 4) Equivalent one mass model

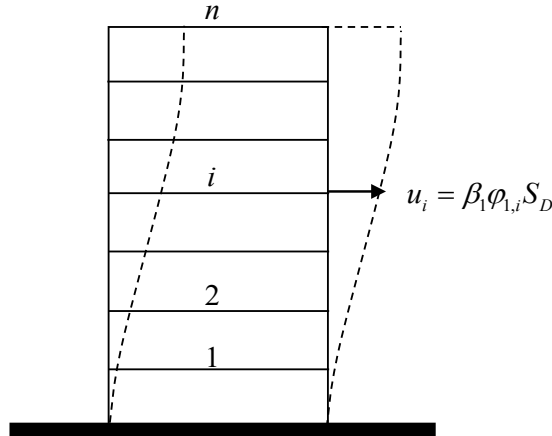
If we consider the first-order mode to be dominant in a multi-story building, the displacement distribution is expressed as,

$$\{u(t)\} = \beta_1 \{\phi_1\} x_1(t) \quad (5-3-35)$$

If we consider that the response is harmonic near its maximum value,

$$\{u(t)\} = \beta_1 \{\phi_1\} S_D e^{i\omega_1 t} \quad (5-3-36)$$

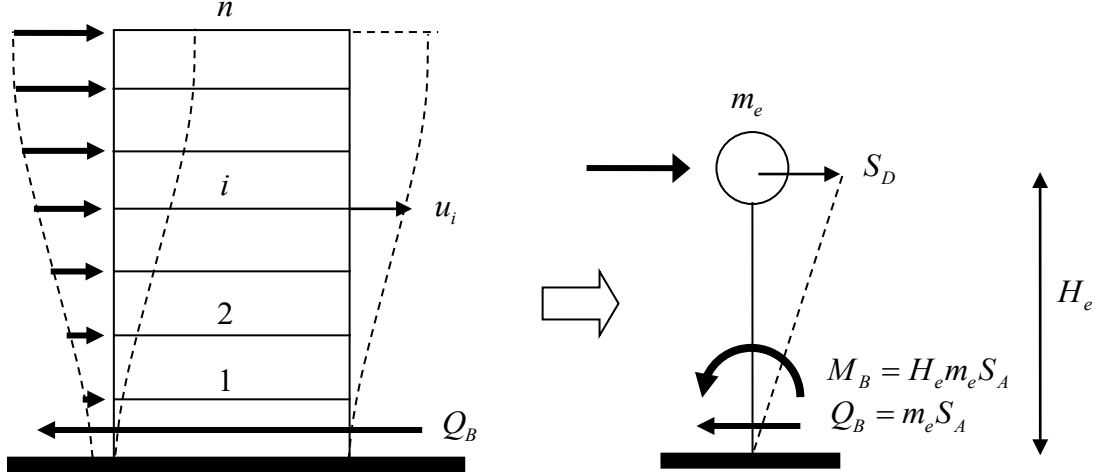
where  $S_D$  is the displacement response spectrum of the first mode.



The acceleration response is

$$\{\ddot{u}(t)\} = (-\omega_1^2) \beta_1 \{\phi_1\} S_D e^{i\omega_1 t} = -\beta_1 \{\phi_1\} S_A e^{i\omega_1 t} \quad (5-3-37)$$

where  $S_A = \omega_1^2 S_D$  is the pseudo acceleration response spectrum of the first mode.



Besides the energy-based definition, **the effective mass**,  $m_e$ , can be defined from the condition that the base shear,  $Q_B$ , is equal in the two models. The base shear of the multi-story building balanced by the inertia forces of the upper floors is

$$Q_B = -\sum_{i=1}^n m_i \ddot{u}_i(t) = \sum_{i=1}^n m_i (\beta_1 S_A \phi_{1,i}) = \beta_1 \left( \sum_{i=1}^n m_i \phi_{1,i} \right) S_A \quad (5-3-38)$$

The base shear of the SDOF system is,

$$Q_B = m_{e,1} S_A \quad (5-3-39)$$

Therefore,

$$m_{e,1} = \beta_1 \left( \sum_{i=1}^n m_i \phi_{1,i} \right) = \frac{\left( \sum_{i=1}^n m_i \phi_{1,i} \right)^2}{\sum_{i=1}^n m_i \phi_{1,i}^2} \quad (5-3-40)$$

**The effective height**,  $H_e$ , is obtained from the condition that overturning moment at the base of the building,  $M_B$ , is equal in the two models. The overturning moment of the multi-story building is

$$M_B = \sum_{i=1}^n m_i \ddot{u}_i(t) H_i = \sum_{i=1}^n m_i H_i (\beta_1 S_A \phi_{1,i}) = \beta_1 \left( \sum_{i=1}^n m_i H_i \phi_{1,i} \right) S_A \quad (5-3-41)$$

The overturning moment of the SDOF system is,

$$M_B = H_{e,1} (m_{e,1} S_A) \quad (5-3-42)$$

Therefore,

$$H_{e,1} = \frac{\beta_1 \left( \sum_{i=1}^n m_i H_i \phi_{1,i} \right)}{m_{e,1}} = \frac{\sum_{i=1}^n m_i H_i \phi_{1,i}}{\sum_{i=1}^n m_i \phi_{1,i}} \quad (5-3-43)$$

The displacement of the representative point is defined as

$$\Delta_d = \frac{\sum_{i=1}^n m_i u_i^2}{\sum_{i=1}^n m_i u_i} = \frac{\sum_{i=1}^n m_i (\beta_1 \phi_{1,i} S_D)^2}{\sum_{i=1}^n m_i (\beta_1 \phi_{1,i} S_D)} = \beta_1 \frac{\sum_{i=1}^n m_i \phi_{1,i}^2}{\sum_{i=1}^n m_i \phi_{1,i}} S_D = S_D \quad (5-3-44)$$

It is consistent with the displacement response spectrum.

Also, the acceleration response spectrum is obtained as

$$S_A = \frac{Q_B}{m_{e,1}} = \frac{\sum_{i=1}^n m_i \phi_{1,i}^2}{\left( \sum_{i=1}^n m_i \phi_{1,i} \right)^2} Q_B \quad (5-3-45)$$

## 5) Initial condition

The initial conditions are obtained from Equation (5-3-10) as,

$$[\Phi]^T [M] \{u(t)\} = [\Phi]^T [M] [\Phi] \{q(t)\} = [\bar{M}] \{q(t)\} \quad (5-3-46)$$

Therefore,

$$\{q\}_{t=0} = [\bar{M}]^{-1} [\Phi]^T [M] \{u\}_{t=0}, \quad \{\dot{q}\}_{t=0} = [\bar{M}]^{-1} [\Phi]^T [M] \{\dot{u}\}_{t=0} \quad (5-3-47)$$

## 5.4 Damping matrix

### Reference:

T. K. Caughey, M. E. J. O'Kelly, "Classical Normal Modes in Damped Linear Dynamic Systems", J. Appl. Mech. Sep 1965, 32(3): 583-588.

### 1) Proportional damping

The general form of the proportional damping is given by Caughey as follows:

$$[C] = [M] \left\{ a_0 + a_1 [M]^{-1} [K] + a_2 \left( [M]^{-1} [K] \right)^2 + \dots + a_{N-1} \left( [M]^{-1} [K] \right)^{N-1} \right\} \quad (5-4-1)$$

From the eigen problem  $[K] \{\varphi_i\} = \omega_i^2 [M] \{\varphi_i\}$ ,

$$[M]^{-1} [K] \{\varphi_i\} = \omega_i^2 \{\varphi_i\} \quad (5-4-2)$$

Substituting into (5-4-1),

$$\begin{aligned} [C] \{\varphi_i\} &= [M] \left\{ a_0 + a_1 [M]^{-1} [K] + a_2 \left( [M]^{-1} [K] \right)^2 + \dots + a_{N-1} \left( [M]^{-1} [K] \right)^{N-1} \right\} \{\varphi_i\} \\ &= \left\{ a_0 + a_1 \omega_i^2 + a_2 \left( \omega_i^2 \right)^2 + \dots + a_{N-1} \left( \omega_i^2 \right)^{N-1} \right\} [M] \{\varphi_i\} \end{aligned} \quad (5-4-3)$$

Therefore, the damping matrix is diagonalized by  $[\Phi]$  as

$$[\Phi]^T [C] [\Phi] = [\bar{C}] = \begin{bmatrix} \bar{c}_1 & & & \\ & \bar{c}_2 & & \\ & & \ddots & \\ & & & \bar{c}_n \end{bmatrix} \quad (5-4-4)$$

$$\bar{c}_i = \left\{ a_0 + a_1 \omega_i^2 + a_2 \left( \omega_i^2 \right)^2 + \dots + a_{N-1} \left( \omega_i^2 \right)^{N-1} \right\} \bar{m}_i \quad (5-4-5)$$

The damping factor is obtained as

$$h_i = \frac{\bar{c}_i}{2 \omega_i \bar{m}_i} = \frac{1}{2} \left\{ \frac{a_0}{\omega_i} + a_1 \omega_i + a_2 \omega_i^3 + \dots + a_{N-1} \omega_i^{2N-1} \right\} \quad (5-4-6)$$

When  $a_0 \neq 0$ ,  $a_1 = a_2 = \dots = a_{N-1} = 0$ , from (5-4-1)

$$[C] = a_0 [M] \quad (5-4-7)$$

This is called a **mass-proportional damping**.

From (5-4-6), the damping factor of the mass proportional damping is

$$h_i = \frac{a_0}{2} \frac{1}{\omega_i} \quad (5-4-8)$$

Therefore, the coefficient  $a_0$  can be determined from the damping factor  $h_1$  of the first mode as

$$a_0 = 2h_1\omega_1$$

When  $a_1 \neq 0$ ,  $a_0 = a_2 = a_3 = \dots = a_{N-1} = 0$ , from (5-4-1)

$$[C] = a_1 [K] \quad (5-4-9)$$

This is called a **stiffness-proportional damping**.

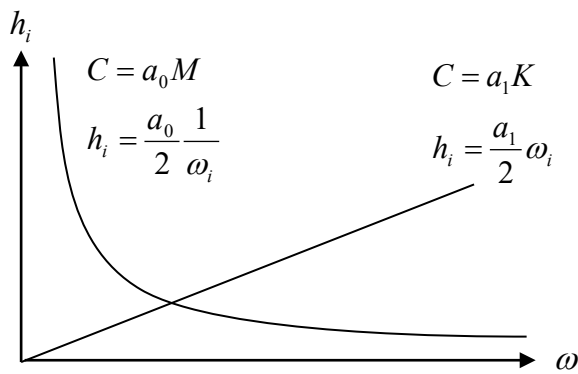
From (5-4-6), the damping factor of the stiffness proportional damping is

$$h_i = \frac{a_1}{2} \omega_i \quad (5-4-10)$$

Therefore, the coefficient  $a_1$  can be determined from the damping factor  $h_1$  of the first mode as

$$a_1 = \frac{2h_1}{\omega_1} \quad (5-4-11)$$

As you can see in the figure below, the damping factor in mass-proportional damping becomes smaller for higher-order modes. On the other hand, the damping factor in stiffness-proportional becomes larger for higher-order modes.



When  $a_0 \neq 0$ ,  $a_1 \neq 0$ ,  $a_2 = a_3 = \dots = a_{N-1} = 0$ , from (5-4-1)

$$[C] = a_0 [M] + a_1 [K] \quad (5-4-12)$$

This is called a **Rayleigh damping**.

From (5-4-6), the damping factor of the Rayleigh damping is

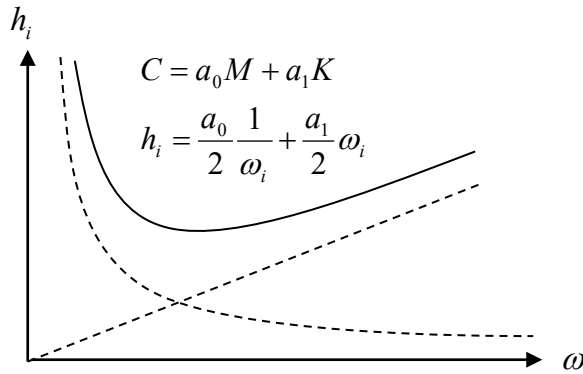
$$h_i = \frac{a_0}{2} \frac{1}{\omega_i} + \frac{a_1}{2} \omega_i \quad (5-4-13)$$

The coefficients  $a_0, a_1$  can be determined from the damping factors  $h_1, h_2$  of the first and second modes. Expressing (5-4-13) for these two modes in matrix form leads to:

$$\begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1/\omega_1 & \omega_1 \\ 1/\omega_2 & \omega_2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \quad (5-4-14)$$

Solving the above equation, we obtain the coefficients  $a_0, a_1$ :

$$\begin{aligned} a_0 &= \frac{2\omega_1\omega_2(\omega_1 h_2 - \omega_2 h_1)}{(\omega_1^2 - \omega_2^2)} \\ a_1 &= \frac{2(\omega_1 h_2 - \omega_2 h_1)}{(\omega_1^2 - \omega_2^2)} \end{aligned} \quad (5-4-15)$$



In the case of mass proportional damping, the size of the damping matrix is the same as the mass matrix, so only **the diagonal components need to be stored**. In the case of stiffness-proportional damping and Rayleigh damping, the size of the stiffness matrix is the same as the stiffness matrix, so the memory size can be reduced using **the skyline method**.

If you want to specify the damping factors up to the  $L$ -th mode ( $L \leq N$ ), by setting  $a_L = a_{L+1} = \dots = a_{N-1} = 0$  in (5-4-6)

$$h_i = \frac{1}{2} \left\{ \frac{a_0}{\omega_i} + a_1 \omega_i + a_2 \omega_i^2 + \dots + a_{L-1} \omega_i^{2L-1} \right\} \quad (5-4-16)$$

In a matrix form

$$\begin{Bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1/\omega_1 & \omega_1 & \dots & \omega_1^{2L-1} \\ 1/\omega_2 & \omega_2 & \dots & \omega_2^{2L-1} \\ \vdots & \vdots & & \vdots \\ 1/\omega_L & \omega_L & \dots & \omega_L^{2L-1} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{L-1} \end{Bmatrix} \quad (5-4-17)$$

Solving the above equation, we obtain the coefficients  $a_0, a_1, \dots, a_{L-1}$ .

From (5-4-4), the damping matrix is obtained as

$$[C] = ([\Phi]^T)^{-1} [\bar{C}] ([\Phi])^{-1} = ([\Phi]^T)^{-1} \begin{bmatrix} \bar{c}_1 & & & \\ & \bar{c}_2 & & \\ & & \ddots & \\ & & & \bar{c}_n \end{bmatrix} ([\Phi])^{-1} \quad (5-4-18)$$

where

$$\bar{c}_i = \left\{ a_0 + a_1 \omega_i^2 + a_2 (\omega_i^2)^2 + \dots + a_{L-1} (\omega_i^2)^{L-1} \right\} \bar{m}_i \quad (5-4-19)$$

$$\text{Substituting } ([\Phi]^T)^{-1} = [M][\Phi] \begin{bmatrix} 1/\bar{m}_1 & & & \\ & 1/\bar{m}_2 & & \\ & & \ddots & \\ & & & 1/\bar{m}_n \end{bmatrix}, \quad (5-4-20)$$

$$[C] = [X] \begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{bmatrix} [X]^T \quad (5-4-21)$$

$$\text{where } [X] = [M][\Phi], \quad b_i = \frac{\bar{c}_i}{\bar{m}_i^2} = \frac{2h_i \omega_i}{\bar{m}_i} \quad (5-4-22)$$

This is called a **Modal damping**.

In the case of modal damping, where the order is 3 or higher, the damping matrix is a **full matrix** and requires a larger memory size than the mass matrix or stiffness matrix. For this reason, STERA\_3D does not employ modal damping.

## 2) Instantaneous stiffness proportional damping

In STERA\_3D, there are two types of stiffness-proportional damping.

One is the proportional damping using the initial stiffness matrix:

$$[C] = \frac{2h}{\omega_1} [K_0] \quad (5-4-23)$$

where, h: damping factor,  $\omega_1$ : circular frequency of the first natural mode,  $[K_0]$ : the initial stiffness.

Another is the proportional damping using the instantaneous stiffness matrix

$$[C] = \frac{2h}{\omega} [K_p] \quad (5-4-24)$$

where, h: damping factor,  $\omega_1$ : circular frequency of the first natural mode,  $[K_p]$ : the instantaneous stiffness changing according to the nonlinearity of structural elements.

In the scene of the practical design of Japan, it is common to use the proportional damping using the instantaneous stiffness matrix.

## 3) Damping matrix with a base isolation building

In the actual design practice for the base isolation buildings, it is common to assume zero viscous damping for horizontal components of the base isolation floor. For example, in case of the stiffness-proportional damping, the damping matrix is defined as:

$$[C] = \frac{2h}{\omega} ([K_{upper}] + [K_{BI,V}]) \quad (5-4-25)$$

where,

$[K_{upper}]$ : the stiffness matrix consisted with upper structures without base isolation elements,

$[K_{BI,V}]$ : the stiffness matrix of base isolation elements for vertical components.

## 4) Damping matrix with viscous damper devices

If there are some viscous damper devices in a structure, in addition to the proportional damping matrix, the global damping matrix formulated from element damping matrices are considered as:

$$[C] = [C_{pro}] + [C_v] \quad (5-4-26)$$

where,  $[C_{pro}]$ : the proportional damping matrix,  $[C_v]$ : the global damping matrix formulated from element damping matrices in the same manner of the global stiffness matrix.

## 5.5 Input ground acceleration

Earthquake ground motions are defined as three components acceleration;  $\ddot{X}_0$ ,  $\ddot{Y}_0$  and  $\ddot{Z}_0$ , in X, Y and Z directions. The inertia forces at node i are defined as,

$$\begin{Bmatrix} \vdots \\ \vdots \\ -M_i(\ddot{u}_{xi} + \ddot{X}_0) \\ -M_i(\ddot{u}_{yi} + \ddot{Y}_0) \\ -M_i(\ddot{\delta}_{zi} + \ddot{Z}_0) \\ -I_i\ddot{\theta}_{xi} \\ -I_i\ddot{\theta}_{yi} \\ -I_i\ddot{\theta}_{zi} \\ \vdots \\ \vdots \end{Bmatrix} = -[M] \begin{Bmatrix} \vdots \\ \vdots \\ \ddot{u}_{xi} \\ \ddot{u}_{yi} \\ \ddot{\delta}_{zi} \\ \ddot{\theta}_{xi} \\ \ddot{\theta}_{yi} \\ \ddot{\theta}_{zi} \\ \vdots \\ \vdots \end{Bmatrix} - [M] \begin{bmatrix} \vdots \\ \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots \\ \vdots \end{bmatrix} \begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 \end{Bmatrix} = -[M] \begin{Bmatrix} \vdots \\ \vdots \\ \ddot{u}_{xi} \\ \ddot{u}_{yi} \\ \ddot{\delta}_{zi} \\ \ddot{\theta}_{xi} \\ \ddot{\theta}_{yi} \\ \ddot{\theta}_{zi} \\ \vdots \\ \vdots \end{Bmatrix} - [M][U] \begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 \end{Bmatrix} \quad (5-5-1)$$

For example, the components of the matrix  $[U]$  of the structure in Figure 5-5-1 will be as follows:

	$\ddot{X}_0$	$\ddot{Y}_0$	$\ddot{Z}_0$
Node 6	0	0	1
	0	0	0
Node 7	0	0	1
	0	0	0
Node 8	0	0	1
	0	0	0
	0	0	0
Node 9	0	0	1
	0	0	0
	0	0	0
Node 10	1	0	0
	0	1	0
	0	0	0

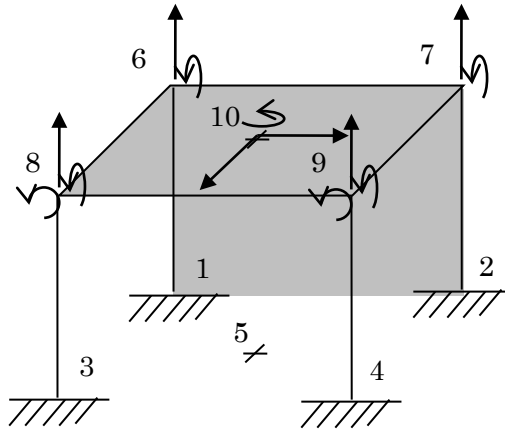


Figure 5-5-1 Components of the matrix  $[U]$

Equilibrium condition of the structure under earthquake ground motion is:

$$\begin{array}{c}
 \underbrace{[C]\{\dot{u}\}}_{\text{Damping force}} + \underbrace{[K]\{u\}}_{\text{Restoring force}} = -\underbrace{[M]\{\ddot{u}\} - [M][U]\begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 \end{Bmatrix}}_{\text{Inertia force}}
 \end{array} \quad (5-5-2)$$

Finally the equation of motion is obtained as:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = -[M][U]\begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 \end{Bmatrix} = \{P\} \quad (5-5-3)$$

## 5.6 External force by vibrator

A vibrator is assumed to be located at the center of gravity at a certain floor. The external forces from the vibrator are denoted as  $F_x, F_y$  in X and Y directions.

$$\begin{Bmatrix} \vdots \\ \vdots \\ \vdots \\ F_x \\ F_y \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix} = \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = [V] \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \quad (5-6-1)$$

For example, the components of the matrix  $[V]$  of the structure in Figure 5-6-1 will be as follows:

	$F_x$	$F_y$
Node 6	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	
Node 7	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	
Node 8	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	
Node 9	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	
Node 10	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$	

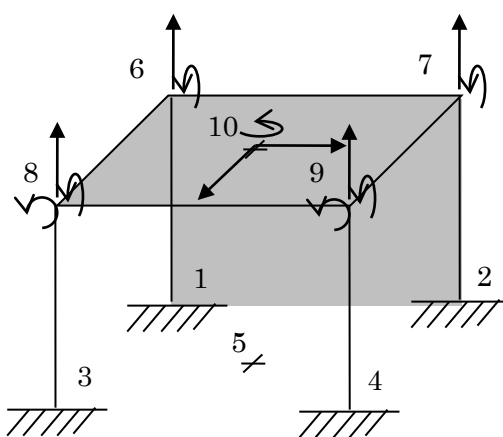


Figure 5-6-1 Components of the matrix  $[V]$

Equilibrium condition of the structure under vibrator force is:

$$\begin{array}{c}
 \underbrace{[C]\{\dot{u}\}}_{\text{Damping force}} + \underbrace{[K]\{u\}}_{\text{Restoring force}} = -\underbrace{[M]\{\ddot{u}\}}_{\text{Inertia force}} + \underbrace{[U]\begin{Bmatrix} F_x \\ F_y \end{Bmatrix}}_{\text{External force}}
 \end{array}
 \quad (5-6-2)$$

Finally the equation of motion is obtained as:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = [U]\begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \{P\}
 \quad (5-6-3)$$

## 5.7 External force by wind

A wind force is assumed to be applied at the center of gravity at each floor with the constant distribution along the height of the building. The external forces at  $i$ -th floor from the wind are denoted as  $h_{i,x}F_x(t)$ ,  $h_{i,y}F_y(t)$ ,  $h_{r,i}M_z(t)$  in X, Y horizontal directions and Z rotational direction.

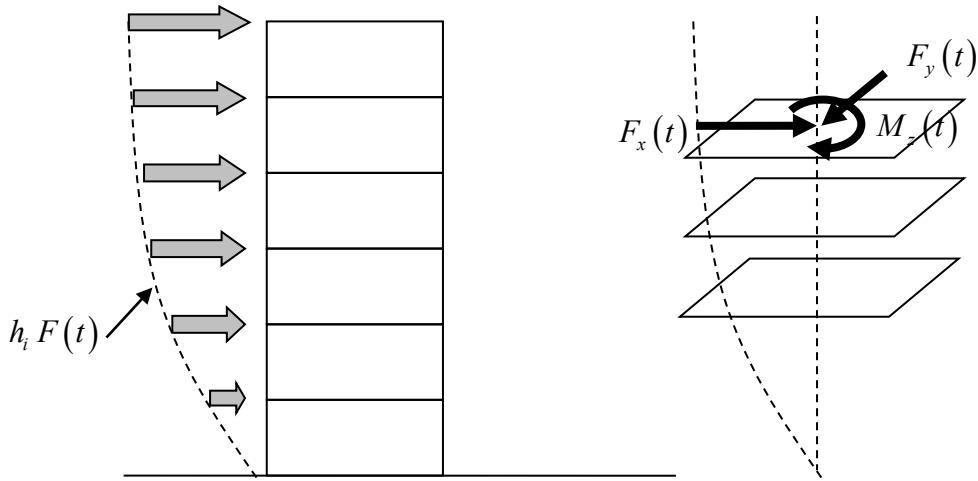


Figure 5-7-1 Wind force distribution

$$\begin{Bmatrix} \vdots \\ \vdots \\ h_{x,i}F_x(t) \\ h_{y,i}F_y(t) \\ 0 \\ 0 \\ 0 \\ h_{r,i}M_z(t) \\ \vdots \\ \vdots \end{Bmatrix} = [W] \begin{Bmatrix} F_x(t) \\ F_y(t) \\ M_z(t) \end{Bmatrix}, \quad [W] = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ h_{x,i} & 0 & 0 \\ 0 & h_{y,i} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{r,i} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \quad (5-7-1)$$

For example, the components of the matrix  $[W]$  of the structure in Figure 5-7-1 will be as follows:

	$F_x$	$F_y$	$M_z$
Node 6	0	0	0
	0	0	0
Node 7	0	0	0
	0	0	0
Node 8	0	0	0
	0	0	0
	0	0	0
Node 9	0	0	0
	0	0	0
	0	0	0
Node 10	$h_{x,1}$	0	0
	0	$h_{y,1}$	0
	0	0	$h_{r,1}$

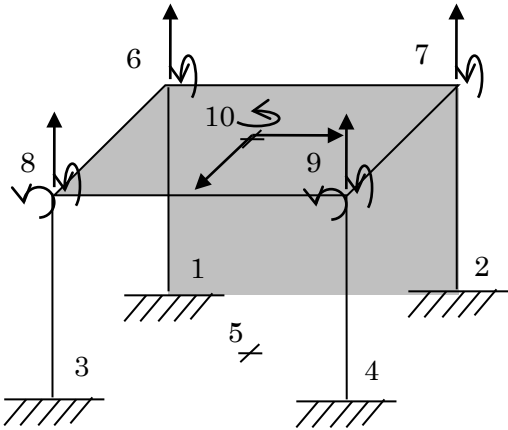


Figure 5-7-2 Components of the matrix  $[W]$

Equilibrium condition of the structure under wind force is:

$$\underbrace{[C]\{\dot{u}\}}_{\text{Damping force}} + \underbrace{[K]\{u\}}_{\text{Restoring force}} = -\underbrace{[M]\{\ddot{u}\}}_{\text{Inertia force}} + \underbrace{[W]\begin{Bmatrix} F_x \\ F_y \\ W_z \end{Bmatrix}}_{\text{External force}} \quad (5-7-2)$$

Finally the equation of motion is obtained as:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = [W]\begin{Bmatrix} F_x \\ F_y \\ W_z \end{Bmatrix} = \{P\} \quad (5-7-3)$$

## 5.8 Numerical integration method

Two numerical integration methods are prepared; one is the Newmark- $\beta$  method with incremental formulation using a step-by-step stiffness matrix, and another one is the Force correction method using a step-by-step stiffness and a force vector together. In case it is difficult to define the step-by-step stiffness of the element such as the case of using a viscous damper element, the Operator Splitting method is selected.

### a) Equation of motion and its incremental form

The equation of motion of a structural system is written as,

$$[M]\{a\} + [C]\{v\} + [K]\{d\} = \{p\} \quad (5-8-1)$$

where,  $[M]$ ,  $[C]$  and  $[K]$  are the mass, damping and stiffness matrices.  $\{d\}$ ,  $\{v\}$ ,  $\{a\}$  and  $\{p\}$  are the displacement, velocity, acceleration, and external force vectors.

The incremental formulation for the equation of motion is,

$$[M]\{\Delta a_i\} + [C]\{\Delta v_i\} + [K]\{\Delta d_i\} = \{\Delta p_i\} \quad (5-8-2)$$

where,  $\{\Delta d_i\}$ ,  $\{\Delta v_i\}$ ,  $\{\Delta a_i\}$  and  $\{\Delta p_i\}$  are the increments of the displacement, velocity, acceleration, and external force vectors, that is,

$$\{\Delta d_i\} \equiv \{d_{i+1}\} - \{d_i\}, \quad \{\Delta v_i\} \equiv \{v_{i+1}\} - \{v_i\}, \quad \{\Delta a_i\} \equiv \{a_{i+1}\} - \{a_i\}, \quad \{\Delta p_i\} \equiv \{p_{i+1}\} - \{p_i\} \quad (5-8-3)$$

In case of a system with hysteresis nonlinearity, the equation of motion can be described as,

$$[M]\{a\} + [C]\{v\} + \{f(d)\} = \{p\} \quad (5-8-4)$$

where  $f(d)$  is the force as a nonlinear function of the displacement  $\{d\}$ . The incremental form can be,

$$[M]\{\Delta a_i\} + [C]\{\Delta v_i\} + \{\Delta f_i(d)\} = \{\Delta p_i\} \quad (5-8-5)$$

In a small time-increment, it can be assumed as a linear relationship in force-deformation as shown in Figure 5-8-1,

$$\{\Delta f_i(d)\} = [K_i]\{\Delta d_i\} \quad (5-8-6)$$

Finally, the equation of motion in incremental form is the same as Equation (5-8-2), that is

$$[M]\{\Delta a_i\} + [C]\{\Delta v_i\} + [K_i]\{\Delta d_i\} = \{\Delta p_i\} \quad (5-8-7)$$

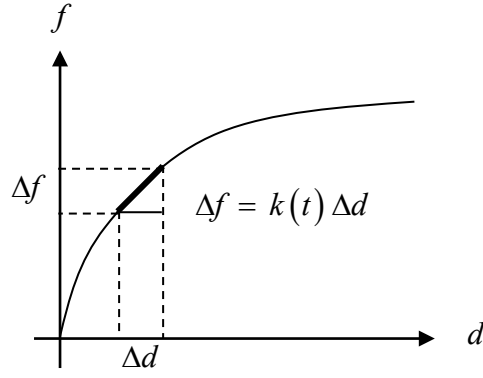


Figure 5-8-1 Nonlinear force-deformation relationship

In the initial condition, the building will deform under the gravity load, i.e., the dead and live loads. It can be analyzed by solving the following equation,

$$[M]\{a\} + [C]\{v\} + [K]\{d\} = -[M][U] \begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 - g \end{Bmatrix} \quad (5-8-26)$$

where  $g$  is the gravity acceleration. When the gravitational acceleration is initially applied, the response may fluctuate in the beginning. Therefore, it is better to apply the static gravity force  $\{f_0\}$  instead of acceleration as,

$$[M]\{a\} + [C]\{v\} + [K]\{d\} = -[M][U] \begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 \end{Bmatrix} + \{f_0\}, \quad \{f_0\} = -[M][U] \begin{Bmatrix} 0 \\ 0 \\ g \end{Bmatrix} \quad (5-8-27)$$

and set the initial displacement as  $\{d\} = \{d_0\}$ , where  $\{d_0\}$  is the solution of

$$[K]\{d_0\} = \{f_0\}$$

The incremental form will be

$$[M]\{\Delta a_i\} + [C]\{\Delta v_i\} + [K_i]\{\Delta d_i\} = -[M][U] \begin{Bmatrix} \Delta \ddot{X}_0 \\ \Delta \ddot{Y}_0 \\ \Delta \ddot{Z}_0 \end{Bmatrix}$$

## b) Newmark- $\beta$ method

The incremental formulation for the equation of motion of a structural system is,

$$[M]\{\Delta a_i\} + [C]\{\Delta v_i\} + [K]\{\Delta d_i\} - \{\Delta f\} = \{\Delta p_i\} \quad (5-8-1)$$

where,  $[M]$ ,  $[C]$  and  $[K]$  are the mass, damping and stiffness matrices.  $\{\Delta d_i\}$ ,  $\{\Delta v_i\}$ ,  $\{\Delta a_i\}$  and  $\{\Delta p_i\}$  are the increments of the displacement, velocity, acceleration and external force vectors, that is,

$$\{\Delta d_i\} \equiv \{d_{i+1}\} - \{d_i\}, \quad \{\Delta v_i\} \equiv \{v_{i+1}\} - \{v_i\}, \quad \{\Delta a_i\} \equiv \{a_{i+1}\} - \{a_i\}, \quad \{\Delta p_i\} \equiv \{p_{i+1}\} - \{p_i\} \quad (5-8-2)$$

$\{\Delta f\}$  is the unbalanced force vector in the previous step.

Using the Newmark- $\beta$  method,

$$\{d_{i+1}\} = \{d_i\} + \{v_i\}(\Delta t) + \left(\frac{1}{2} - \beta\right)\{a_i\}(\Delta t)^2 + \beta\{a_{i+1}\}(\Delta t)^2 \quad (5-8-3)$$

$$\{v_{i+1}\} = \{v_i\} + \frac{1}{2}(\{a_i\} + \{a_{i+1}\})(\Delta t) \quad (5-8-4)$$

The incremental form is

$$\{\Delta d_i\} = \{v_i\}(\Delta t) + \frac{1}{2}\{a_i\}(\Delta t)^2 + \beta\{\Delta a_i\}(\Delta t)^2 \quad (5-8-5)$$

$$\{\Delta v_i\} = \{a_i\}(\Delta t) + \frac{1}{2}\{\Delta a_i\}(\Delta t) \quad (5-8-6)$$

From Equation (5-8-5), we obtain

$$\{\Delta a_i\} = \frac{1}{\beta(\Delta t)^2}\{\Delta d_i\} - \frac{1}{\beta(\Delta t)}\{v_i\} - \frac{1}{2\beta}\{a_i\} \quad (5-8-7)$$

Substituting Equation (5-8-7) into Equation (5-8-6) gives

$$\{\Delta v_i\} = \frac{1}{2\beta(\Delta t)}\{\Delta d_i\} - \frac{1}{2\beta}\{v_i\} + \left(1 - \frac{1}{4\beta}\right)\{a_i\}(\Delta t) \quad (5-8-8)$$

Equations (5-8-7) and (5-8-8) are substituted into the equation of motion, Equation (5-8-1), and we obtain

$$\begin{aligned} & \{\Delta d_i\} \left[ \frac{1}{\beta(\Delta t)^2}[M] + \frac{1}{2\beta(\Delta t)}[C] + [K] \right] \\ &= \{\Delta p_i\} + [M] \left[ \frac{1}{\beta(\Delta t)}\{v_i\} + \frac{1}{2\beta}\{a_i\} \right] + [C] \left[ \frac{1}{2\beta}\{v_i\} + \left( \frac{1}{4\beta} - 1 \right)\{a_i\}(\Delta t) \right] + \{\Delta f\} \end{aligned} \quad (5-8-9)$$

The equation can be rewritten as,

$$[\hat{K}] \cdot \{\Delta d_i\} = \{\Delta \hat{p}_i\} \quad (5-8-10)$$

where,

$$[\hat{K}] = [K] + \frac{1}{2\beta(\Delta t)}[C] + \frac{1}{\beta(\Delta t)^2}[M] \quad (5-8-11)$$

$$\{\Delta \hat{p}_i\} = \{\Delta p_i\} + [M] \left( \frac{1}{\beta(\Delta t)} \{v_i\} + \frac{1}{2\beta} \{a_i\} \right) + [C] \left[ \frac{1}{2\beta} \{v_i\} + \left( \frac{1}{4\beta} - 1 \right) \{a_i\} (\Delta t) \right] + \{\Delta f\} \quad (5-8-12)$$

### c) Operator Splitting method

The Operator Splitting (OS) method is a type of mixed integration method in which stiffness is divided into linear and nonlinear (Nakashima, 1990). The explicit predictor-corrector method is employed for the integration associated with the nonlinear stiffness, whereas the unconditionally stable Newmark- $\beta$  method is used for the integration associated with linear stiffness. The formulations are described as follows:

Using the Newmark- $\beta$  method,

$$\{d_{i+1}\} = \{d_i\} + \{v_i\}(\Delta t) + \left(\frac{1}{2} - \beta\right)\{a_i\}(\Delta t)^2 + \beta\{a_{i+1}\}(\Delta t)^2 \quad (5-8-13)$$

$$\{v_{i+1}\} = \{v_i\} + \frac{1}{2}(\{a_i\} + \{a_{i+1}\})(\Delta t) \quad (5-8-14)$$

Introducing the predictor displacement  $\{\tilde{d}_{i+1}\}$  as,

$$\{\tilde{d}_{i+1}\} = \{d_i\} + \{v_i\}(\Delta t) + \left(\frac{1}{2} - \beta\right)\{a_i\}(\Delta t)^2 \quad (5-8-15)$$

Equation (5-8-13) can be written as

$$\{d_{i+1}\} = \{\tilde{d}_{i+1}\} + \beta\{a_{i+1}\}(\Delta t)^2 \quad (5-8-16)$$

Therefore

$$\{a_{i+1}\} = \frac{1}{\beta(\Delta t)^2}(\{d_{i+1}\} - \{\tilde{d}_{i+1}\}) = \frac{1}{\beta(\Delta t)^2}\{\Delta d_{i+1}\} \quad (5-8-17)$$

where

$$\{\Delta d_{i+1}\} = \{d_{i+1}\} - \{\tilde{d}_{i+1}\} \quad (5-8-18)$$

Substituting Equation (5-8-17) into Equation (5-8-14),

$$\{v_{i+1}\} = \frac{1}{2\beta(\Delta t)}\{\Delta d_{i+1}\} + \{v_i\} + \frac{1}{2}\{a_i\}(\Delta t) \quad (5-8-19)$$

In the equation of motion,

$$[M]\{a_{i+1}\} + [C]\{v_{i+1}\} + \{f(d_{i+1})\} = \{p_{i+1}\} \quad (5-8-20)$$

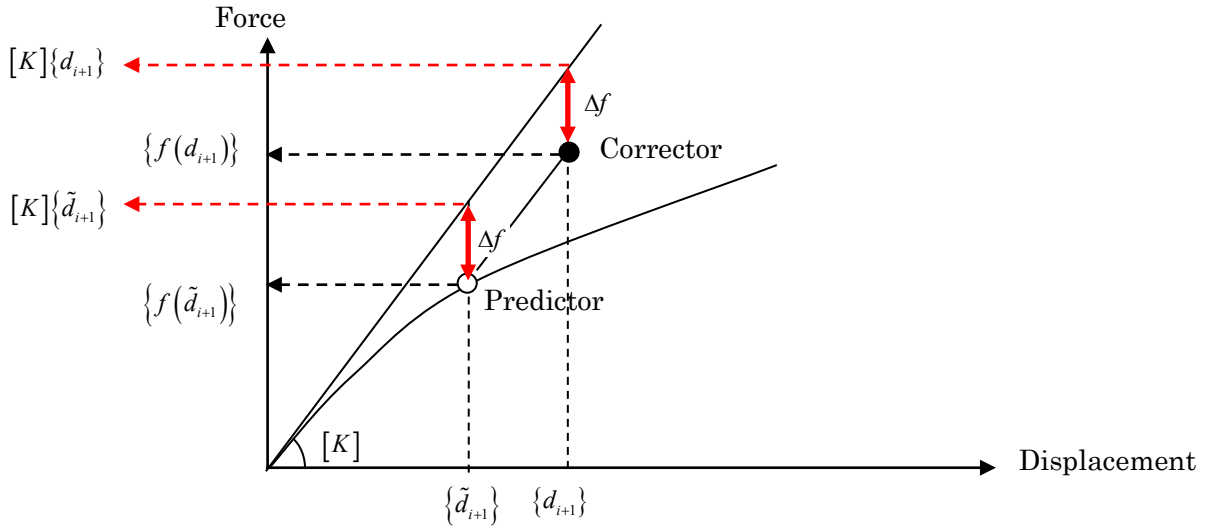
The nonlinear internal resisting forces are approximated as follows:

$$\{f(d_{i+1})\} = [K]\{d_{i+1}\} - \{\Delta f\} \quad (5-8-21)$$

where

$$\{\Delta f\} = [K]\{\tilde{d}_{i+1}\} - \{f(\tilde{d}_{i+1})\} \quad (5-8-22)$$

In this formulation,  $[K]$  is the initial stiffness matrix.

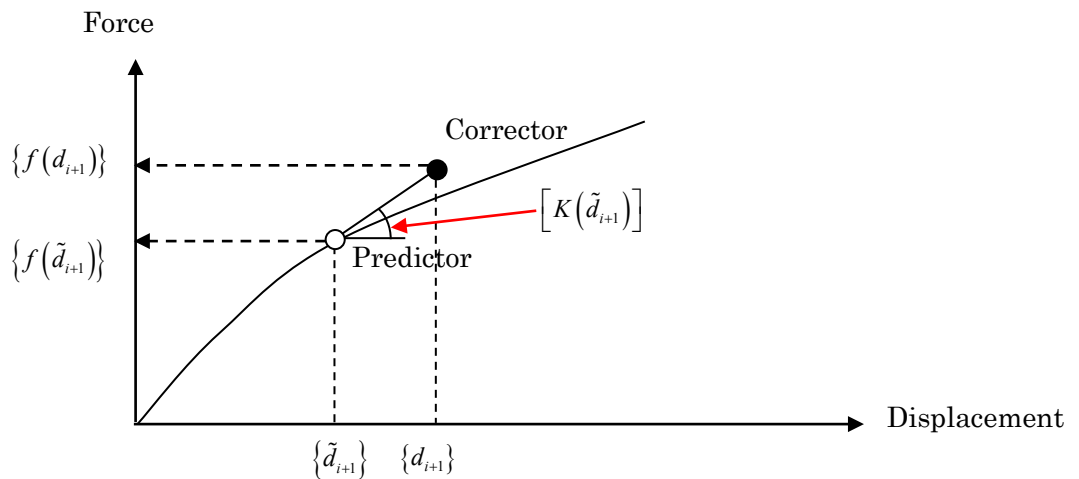


The nonlinear internal resisting forces can be written as,

$$\{f(d_{i+1})\} = [K(\tilde{d}_{i+1})]\{\Delta d_{i+1}\} + \{f(\tilde{d}_{i+1})\} \quad (5-8-23)$$

where  $[K(\tilde{d}_{i+1})]$  is a predictor stiffness.

The predictor stiffness is not necessary to be the initial stiffness and if the predictor stiffness is close to the tangent stiffness, the corrector force is more accurate. It is known that if the predictor stiffness is larger than the tangent stiffness, the OS method is unconditionally stable.



In STERA\_3D, the predictor stiffness is created from the initial stiffness or tangent stiffness if available.

Substituting the above equations into the equation of motion,

$$[M]\{a_{i+1}\} + [C]\{v_{i+1}\} + \{f(d_{i+1})\} = \{p_{i+1}\} \quad (5-8-24)$$

$$[M]\left(\frac{1}{\beta(\Delta t)^2}\{\Delta d_{i+1}\}\right) + [C]\left(\frac{1}{2\beta(\Delta t)}\{\Delta d_{i+1}\} + \{v_i\} + \frac{1}{2}\{a_i\}(\Delta t)\right) + [K(\tilde{d}_{i+1})]\{\Delta d_{i+1}\} + \{f(\tilde{d}_{i+1})\} = \{p_{i+1}\}$$

Solving for  $\{\Delta d_{i+1}\}$ ,

$$[\hat{K}]\{\Delta d_{i+1}\} = \{\hat{p}\} \quad (5-8-25)$$

where

$$[\hat{K}] = [K(\tilde{d}_{i+1})] + \frac{1}{2\beta(\Delta t)}[C] + \frac{1}{\beta(\Delta t)^2}[M] \quad (5-8-26)$$

$$\{\hat{p}\} = -[C]\left(\{v_i\} + \frac{1}{2}\{a_i\}(\Delta t)\right) - \{f(\tilde{d}_{i+1})\} + \{p_{i+1}\} \quad (5-8-27)$$

The procedure for solving the equation of motion is as follows:

- Step 1.* Calculate the predictor displacement vector  $\{\tilde{d}_{i+1}\}$  by Equation (5-8-15).
- Step 2.* Obtain the restoring force  $\{f(\tilde{d}_{i+1})\}$  in reference to the constitutive model.
- Step 3.* Substitute  $\{f(\tilde{d}_{i+1})\}$  to Equation (5-8-27) and solve the displacement increment  $\{\Delta d_{i+1}\}$  from Equation (5-8-25) and obtain the corrector displacement  $\{d_{n+1}\}$  from Equation (5-8-18).

Under seismic excitation and gravity load, the equation of motion will be,

$$[M]\{a_{i+1}\} + [C]\{v_{i+1}\} + \{f(d_{i+1})\} = -[M][U]\left\{\begin{array}{c} \ddot{X}_{0,i+1} \\ \ddot{Y}_{0,i+1} \\ \ddot{Z}_{0,i+1} - g \end{array}\right\} \quad (5-8-28)$$

The initial displacement as  $\{d\} = \{d_0\}$ , where  $\{d_0\}$  is the solution of

$$[K]\{d_0\} = -[M][U]\left\{\begin{array}{c} 0 \\ 0 \\ g \end{array}\right\} \quad (5-8-29)$$

## 5.9 Energy

### a) Equation of energy

As it was mentioned in Equation (5-5-2), the equation of motion is obtained as:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = -[M][U] \begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 \end{Bmatrix} = \{P\} \quad (5-9-1)$$

For example, in case of a structure with a rigid floor in Figure 5-9-1, the displacement vector,  $\{u\}$ , consists of 15 components (see RED numbers in Figure 5-9-1.)

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{15} \end{Bmatrix} \quad (5-9-2)$$

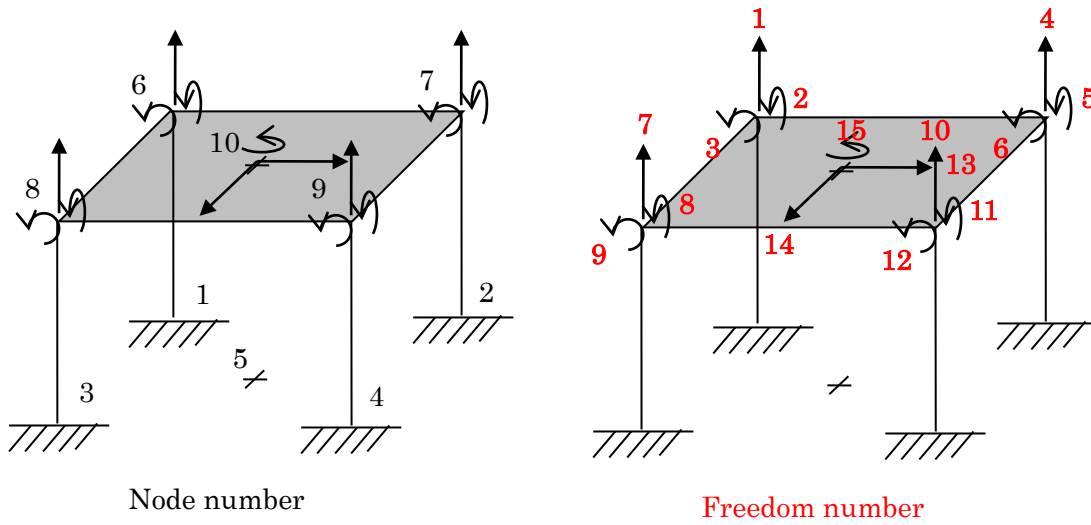


Figure 5-9-1 Example of the freedom vector of a structure with a rigid floor

The equation of energy is derived by multiplying the velocity vector,  $\{\dot{u}\}^T$ , and integrating by the time range  $[0-t]$ :

$$\int_0^t \{\dot{u}\}^T [M] \{\dot{u}\} dt + \int_0^t \{\dot{u}\}^T [C] \{\dot{u}\} dt + \int_0^t \{\dot{u}\}^T [K] \{u\} dt = - \int_0^t \{\dot{u}\}^T \{P\} dt \quad (5-9-3)$$

$$\frac{\{\dot{u}\}^T [M] \{\dot{u}\}}{2} + \int_0^t \{\dot{u}\}^T [C] \{\dot{u}\} dt + \frac{\{u\}^T [K] \{u\}}{2} = - \int_0^t \{\dot{u}\}^T \{P\} dt \quad (5-9-4)$$

$$W_K + W_D + W_P = W_I \quad (5-9-5)$$

where,

$$W_K = \frac{\{\dot{u}\}^T [M] \{\dot{u}\}}{2} \quad : \text{Kinematic energy}$$

$$W_D = \int_0^t \{\dot{u}\}^T [C] \{\dot{u}\} dt \quad : \text{Damping energy}$$

$$W_P = \frac{\{u\}^T [K] \{u\}}{2} \quad : \text{Potential energy}$$

$$W_I = - \int_0^t \{\dot{u}\}^T \{P\} dt \quad : \text{Input energy}$$

If the system is nonlinear, the equation of motion can be expressed as:

$$[M] \{\ddot{u}\} + [C] \{\dot{u}\} + Q(u, \dot{u}) = -[M][U] \left\{ \begin{matrix} \ddot{X}_0 \\ \ddot{Y}_0 \\ \ddot{Z}_0 \end{matrix} \right\} = \{P\} \quad (5-9-6)$$

where,  $Q(u, \dot{u})$  is the nonlinear restoring force vector. Then, the equation of energy can be derived as;

$$W_K + W_D + W_P = W_I \quad (5-9-7)$$

where,

$$W_K = \frac{\{\dot{u}\}^T [M] \{\dot{u}\}}{2} \quad : \text{Kinematic energy}$$

$$W_D = \int_0^t \{\dot{u}\}^T [C] \{\dot{u}\} dt \quad : \text{Damping energy}$$

$$W_P = \int_0^t \{\dot{u}\}^T Q(u, \dot{u}) dt \quad : \text{Potential energy} \quad (5-9-8)$$

$$W_I = - \int_0^t \{\dot{u}\}^T \{P\} dt \quad : \text{Input energy}$$

### b) Decomposition of potential energy

We can decompose the restoring force vector into the restoring force of each member as,

$$Q(u, \dot{u}) = q_1(u, \dot{u}) + q_2(u, \dot{u}) + \cdots + q_n(u, \dot{u}); \quad n : \text{number of members} \quad (5-9-9)$$

Therefore, the potential energy can be decomposed as,

$$W_P = \int_0^t \{\dot{u}\}^T Q(u, \dot{u}) dt = \int_0^t \{\dot{u}\}^T \sum_{i=1}^n q_i(u, \dot{u}) dt = \sum_{i=1}^n \left( \int_0^t \{\dot{u}\}^T q_i(u, \dot{u}) dt \right) = \sum_{i=1}^n W_{P,i} \quad (5-9-10)$$

where

$$W_{P,i} = \int_0^t \{\dot{u}\}^T q_i(u, \dot{u}) dt; \quad \text{potential energy of } i\text{-th member} \quad (5-9-11)$$



## 6. Nonlinear Static Push-Over Analysis

### 6. 1 Lateral distribution of earthquake force

The static lateral load representing the earthquake force is applied at the center of gravity in each floor. There are several formulas to define the load distribution along the height of the building. In “STERA 3D” program, the following distributions are prepared:

1.  $A_i$     2. Triangular    3. Uniform    4. UBC    5. ASCE    6. Mode

#### (1) $A_i$ distribution

In the “Building Standard Law” in Japan, the design shear force of  $i$ -th story,  $Q_i$ , is defined as,

$$Q_i = C_i \sum_{j=i}^n w_j, \quad C_i = Z R_i A_i C_0 \quad (6-1-1)$$

where,

- $C_i$  : design shear coefficient of  $i$ -th story,  
 $w_i$  : weight of  $i$ -th story,  
 $Z$  : seismic zone factor,  
 $R_i$  : vibration characteristic factor taking into consideration of soil condition,  
 $A_i$  : lateral distribution of shear force coefficient,  
 $C_0$  : design base shear coefficient ( $C_0=0.2$  for serviceability limit,  $C_0=1.0$  for safety limit)

If we set,  $Z=1.0$  (Tokyo),  $R_i=1.0$  (stiff soil, a short story building),  $C_0=1.0$  (safety design), the design shear force distribution is simplified as,

$$Q_i = A_i \sum_{j=i}^n w_j \quad (6-1-2)$$

“ $A_i$ ” distribution is defined as,

$$A_i = 1 + \left( \frac{1}{\sqrt{\alpha_i}} - \alpha_i \right) \frac{2T}{1 + 3T} \quad (6-1-3)$$

where,

- $\alpha_i = \sum_{j=i}^n w_j / W$ ,  $W = \sum_{j=1}^n w_j$  : the ratio of weight upper than  $i$ -th story,  
 $T$  : the first natural period of a building ( $=0.02h$ ,  $h$  : the building height)

As shown in Figure 6-1-1, the static lateral load is obtained as,

$$F_n = Q_n, \quad F_i = Q_i - Q_{i+1} \quad (i = 1, \dots, n-1) \quad (6-1-4)$$

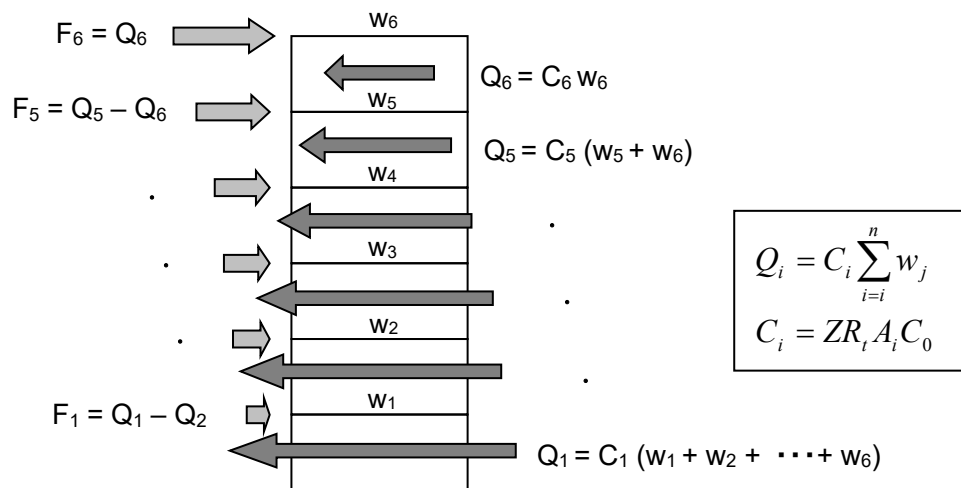


Figure 6-1-1 Ai distribution

## (2) Triangular distribution

Triangular distribution is defined as:

$$F_i = Q_B \left( h_i / \sum_{j=1}^n h_j \right) \quad (6-1-5)$$

where,

$Q_B$  : base shear force

$h_i$  : the height of the i-th story from the ground

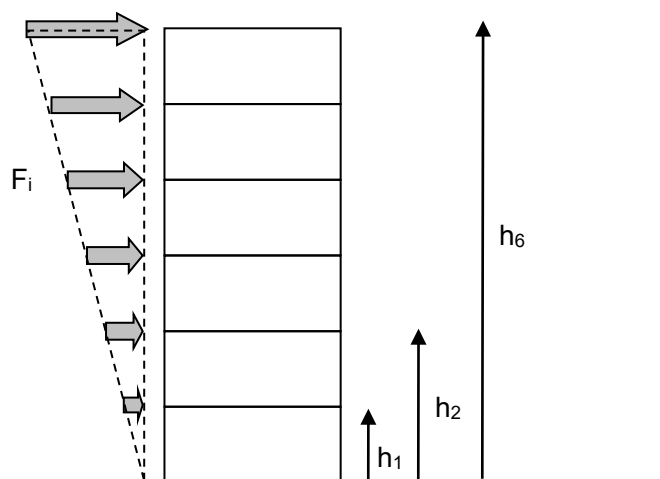


Figure 6-1-2 Triangular distribution

### (3) Uniform distribution

Uniform distribution is defined as:

$$F_i = Q_B(1/n) \quad (6-1-6)$$

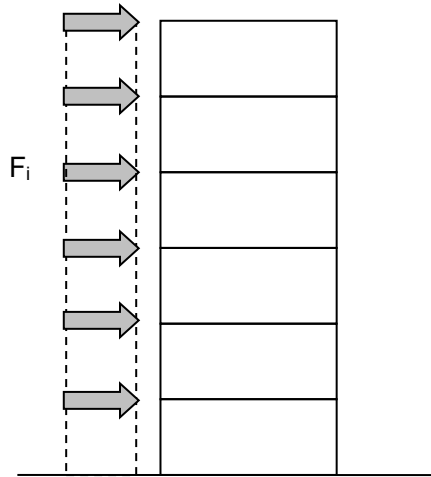


Figure 6-1-3 Uniform distribution

### (4) UBC distribution

The UBC (Uniform Building Code, 1997) gives the following formula for the calculation of lateral force distribution:

$$F_i = (Q_B - F_t) \left( w_i h_i / \sum_{j=1}^n w_j h_j \right) \quad (6-1-7)$$

$$F_t = \begin{cases} 0 & , \text{if } T \leq 0.7 \text{ sec} \\ 0.07 T Q_B & , \text{if } T > 0.7 \text{ sec} \end{cases} \quad (6-1-8)$$

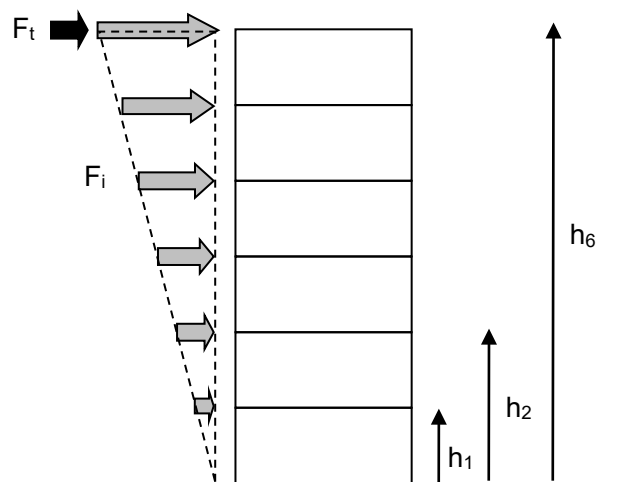


Figure 6-1-4 UBC distribution

### (5) ASCE distribution

The IBC (International Building Code) in the U.S. refers to the ASCE 7 “Seismic Design Requirements for Building Structures” which gives the following formula for the calculation of lateral force distribution:

$$F_i = w_i h_i^k / \sum_{j=1}^n w_j h_j^k \quad (6-1-9)$$

where  $k$  is an exponent related to the structural period as follows:

$$k = \begin{cases} 1 & , \quad \text{if } T < 0.5 \text{ sec} \\ (T - 0.5) / 2 & , \quad \text{if } 0.5 \text{ sec} < T < 2.5 \text{ sec} \\ 2 & , \quad \text{if } T > 2.5 \text{ sec} \end{cases} \quad (6-1-10)$$

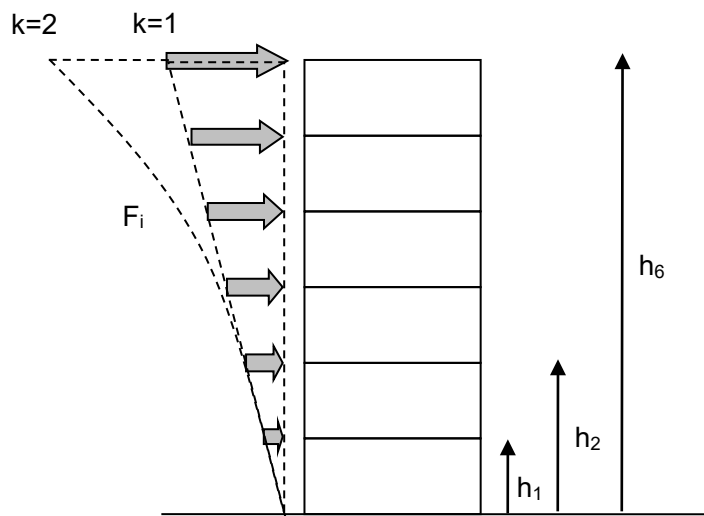


Figure 6-1-5 ASCE distribution

### (6) Mode distribution

Mode distribution is defined as:

$$F_i = Q_B \left( w_i \phi_{1,i} / \sum_{j=1}^n w_j \phi_{1,j} \right) \quad (6-1-11)$$

where,

$\phi_{1,i}$ : component of the first mode distribution in the i-th story

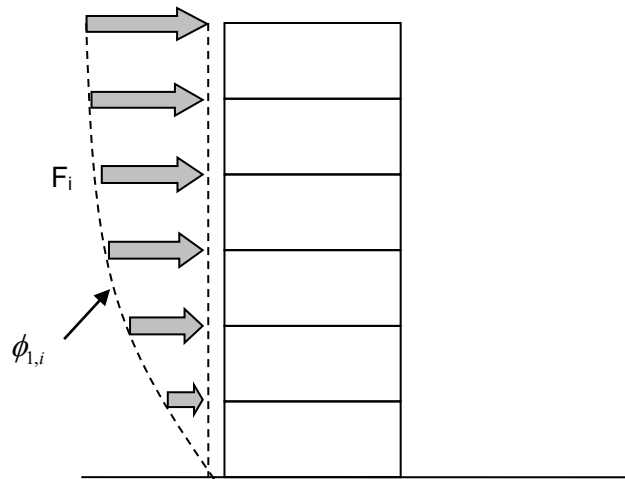


Figure 6-1-6 Mode distribution

## 6. 2 Capacity Curve

The **Capacity Spectrum Method** was proposed by Freeman [1978] as an approximate way to estimate the maximum response of a structure under an earthquake ground motion. The concept was modified by Kuramoto et.al [2000] to adopt the distribution of nonlinear story displacement as the first mode shape in each calculation step. The method was adopted as one of the evaluation procedures in the Building Standard Law, Japan.

The key concept of the Capacity Spectrum Method is to find out the intersection between the **Demand Spectra** (= relationship between  $S_d$  (displacement spectra) and  $S_a$  (acceleration spectra)) and the **Capacity Curve** (= nonlinear push-over curve of an equivalent single-degree-of-freedom system).

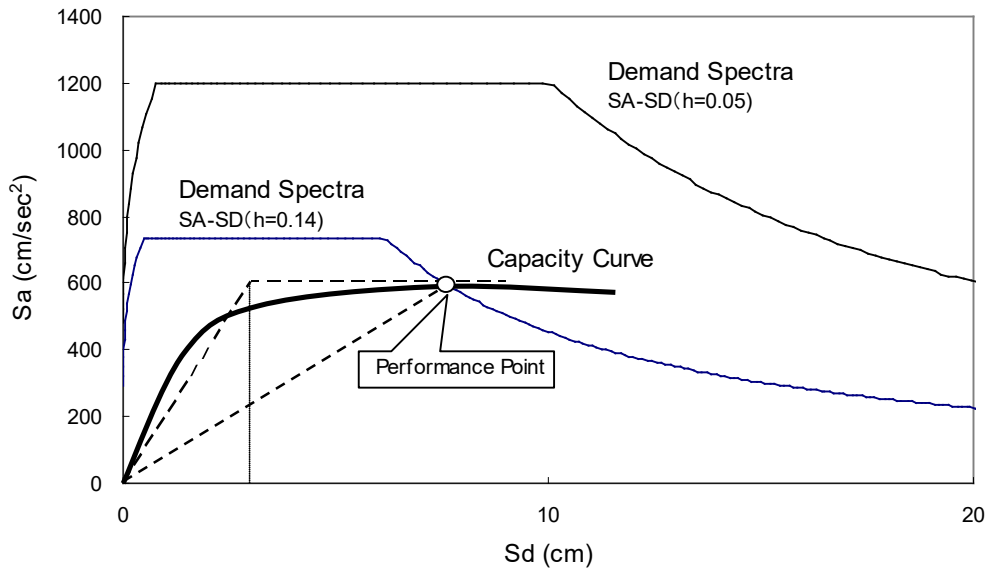


Figure 6-2-1 Schematic example of the concept of Capacity Spectrum Method

As discussed in **5.3 Modal analysis**, if we consider the first-order mode to be dominant in a multi-story building, the displacement and acceleration of the equivalent one mass model are expressed as,

$$S_D = \frac{\sum_{i=1}^n m_i u_i^2}{\sum_{i=1}^n m_i u_i}, \quad S_A = \frac{\sum_{i=1}^n m_i \phi_{1,i}^2}{\left( \sum_{i=1}^n m_i \phi_{1,i} \right)^2} Q_B = \frac{\sum_{i=1}^n m_i u_i^2}{\left( \sum_{i=1}^n m_i u_i \right)^2} Q_B \quad (6-2-1)$$

Representing the displacement by the inelastic rather than the elastic first-mode shape is consistent with characterizing the structure by its secant stiffness to maximum response. Therefore, “STERA 3D” provides the menu in the static analysis to show the Capacity Curve based on the following formula (Kuramoto et.al [2000]):

$$S_D = \frac{\sum_{i=1}^n m_i \Delta_i^2}{\sum_{i=1}^n m_i \Delta_i}, \quad S_A = \frac{\sum_{i=1}^n m_i \Delta_i^2}{\left( \sum_{i=1}^n m_i \Delta_i \right)^2} Q_B \quad (6-2-2)$$

where,

$m_i$  : lumped mass in the i-th story

$\Delta_i$  : component of the distribution of nonlinear story displacement in the i-th story

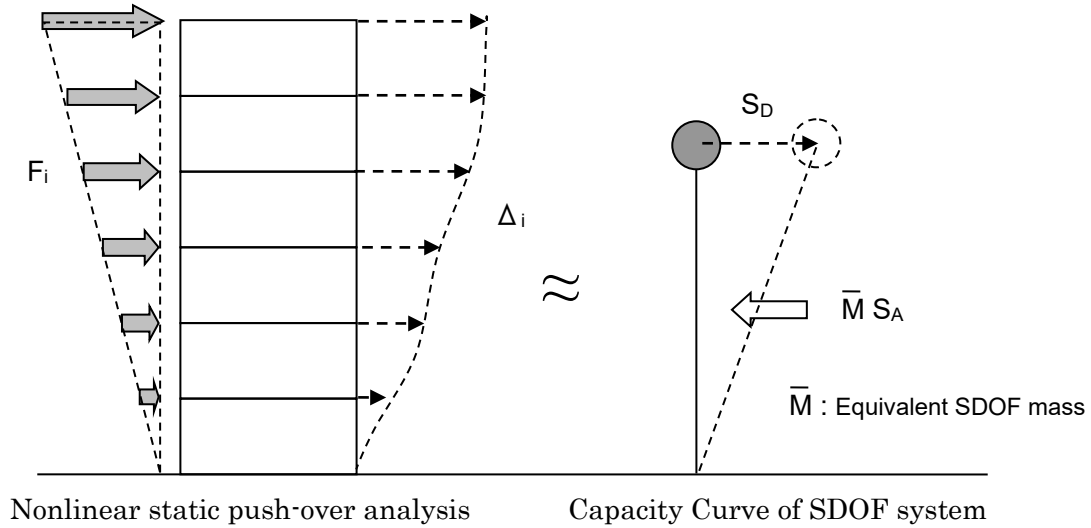


Figure 6-2-2 Capacity Curve of the equivalent SDOF system

As schematically shown in Figure 6-2-2, the step-by-step results of nonlinear push-over analysis is used to obtain the Capacity Curve of the equivalent SDOF system using Equation (6-2-2).

## References

- Freeman S. A. (1978), "Prediction of Response of Concrete Buildings to Severe Earthquake Motion", Douglas McHenry International Symposium on Concrete and Concrete Structures, SP-55, American Concrete Institute, Detroit, Michigan, pp. 589-605.
- Kuramoto H., et.al. (200), "Predicting the Earthquake Response of Buildings using Equivalent Single Degree of Freedom System", 12th World Conference on Earthquake Engineering (12WCEE), Auckland New Zealand, 2000.2.

## 7. Lumped Mass Model

### 7.1 Decomposition of shear and flexural deformation

#### a-1) Equivalent plane for each floor from displacement

The equivalent plane ( $z = ax + by + c$ ) is obtained from the vertical displacement distribution by the least square method:

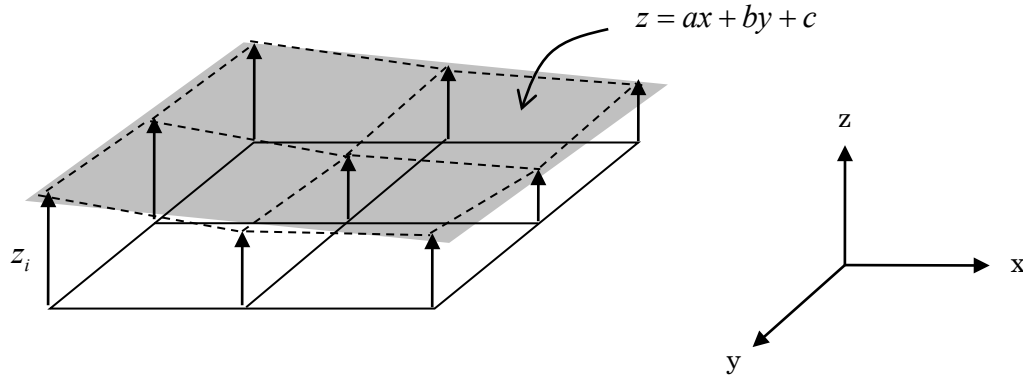


Figure 7-1-1 Equivalent plane

$$\text{Minimize} \quad L = \sum (z_i - (ax_i + by_i + c))^2 \quad (7-1-1)$$

where,  $i$  : node number in the floor

$a, b, c$  : parameters of equivalent plane

$$\text{Thus,} \quad \frac{\partial L}{\partial a} = 0, \quad \frac{\partial L}{\partial b} = 0, \quad \frac{\partial L}{\partial c} = 0 \quad (7-1-2)$$

Parameters,  $a, b, c$  are obtained by solving the following linear equation:

$$\begin{bmatrix} \sum z_i x_i \\ \sum z_i y_i \\ \sum z_i \end{bmatrix} = \begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \text{sym.} & & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (7-1-3)$$

where,

$n$  : the number of nodes in a floor

### a-2) Equivalent plane for each floor from potential energy

The equivalent plane ( $z = ax + by + c$ ) is obtained from the vertical potential energy distribution by the least square method:

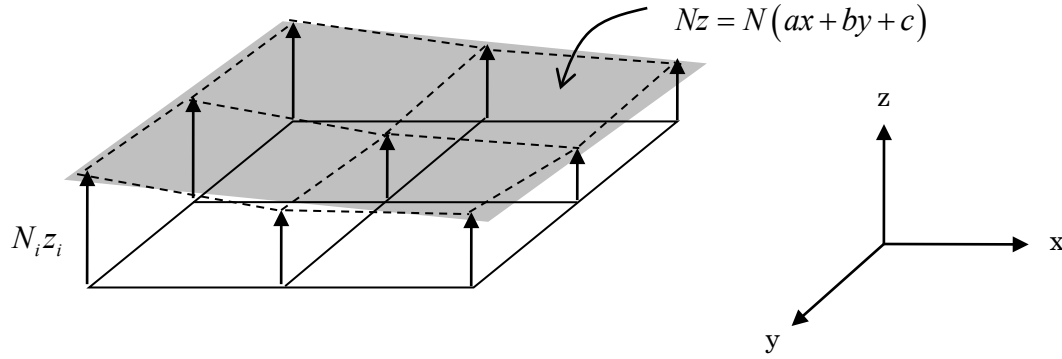


Figure 7-1-2 Equivalent plane

$$\text{Minimize} \quad L = \sum (N_i z_i - N_i (ax_i + by_i + c))^2 \quad (7-1-4)$$

where,  $i$  : node number in the floor

$N_i$  : axial load at node  $i$

$a, b, c$  : parameters of equivalent plane

$$\text{Thus,} \quad \frac{\partial L}{\partial a} = 0, \quad \frac{\partial L}{\partial b} = 0, \quad \frac{\partial L}{\partial c} = 0 \quad (7-1-5)$$

Parameters,  $a, b, c$  are obtained by solving the following linear equation:

$$\begin{bmatrix} \sum N_i z_i x_i \\ \sum N_i z_i y_i \\ \sum N_i z_i \end{bmatrix} = \begin{bmatrix} \sum N_i x_i^2 & \sum N_i x_i y_i & \sum N_i x_i \\ \text{sym.} & \sum N_i y_i^2 & \sum N_i y_i \\ & & \sum N_i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (7-1-6)$$

where,

$n$  : the number of nodes in a floor

At this moment, STERA\_3D adopts the formulation a-1), since it is easier to implement.

## b) Decomposition of shear and flexural deformation

A story drift,  $D$ , can be divided into shear and flexural components as,

$$D = D_S(\text{shear}) + D_F(\text{flexure}) \quad (7-1-7)$$

Assuming the distribution of floor deformation is expressed by an equivalent plane ( $z = ax + by + c$ ), the flexural deformation,  $D_F$ , can be expressed as,

$$D_F = -aH \quad : \text{x-direction} \quad (7-1-8)$$

$$D_F = bH \quad : \text{y-direction} \quad (7-1-9)$$

Note that the coefficient 'a' is the negative angle in x-direction.

Then, the shear deformation can be obtained as,

$$D_S = D - D_F \quad (7-1-10)$$

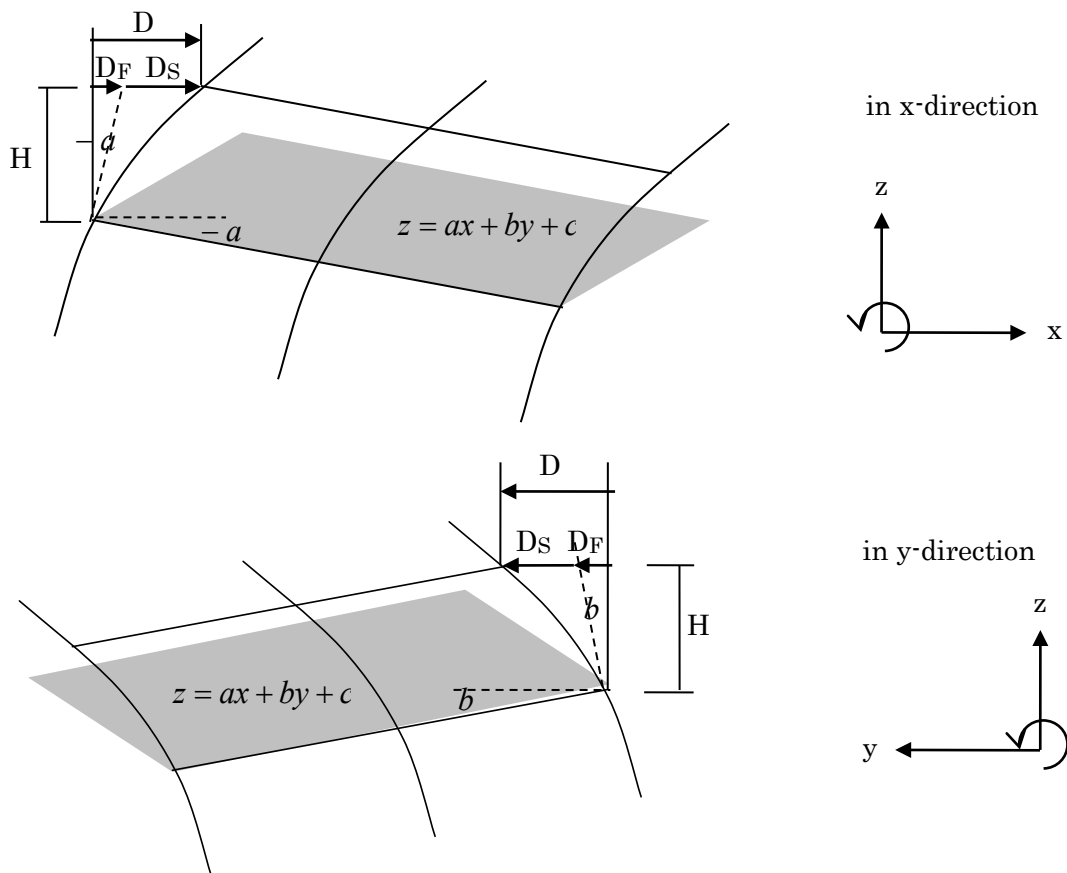


Figure 7-1-3 Decomposition of shear and flexural deformation

## 7.2 Lumped mass model with shear and flexural stiffness

### a) Linear flexural model

The frame model can be idealized as a lumped mass model with a concentrated mass at each floor and shear and flexural springs in each story.

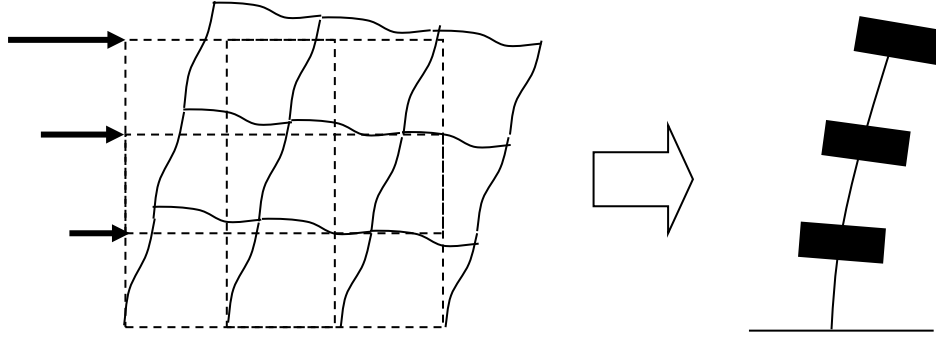


Figure 7-1-4 Idealization to lumped mass model

Under the external lateral forces,  $F_i$  ( $i = 1, 2, 3$ ), the shear force and moment of each story are expressed as below.

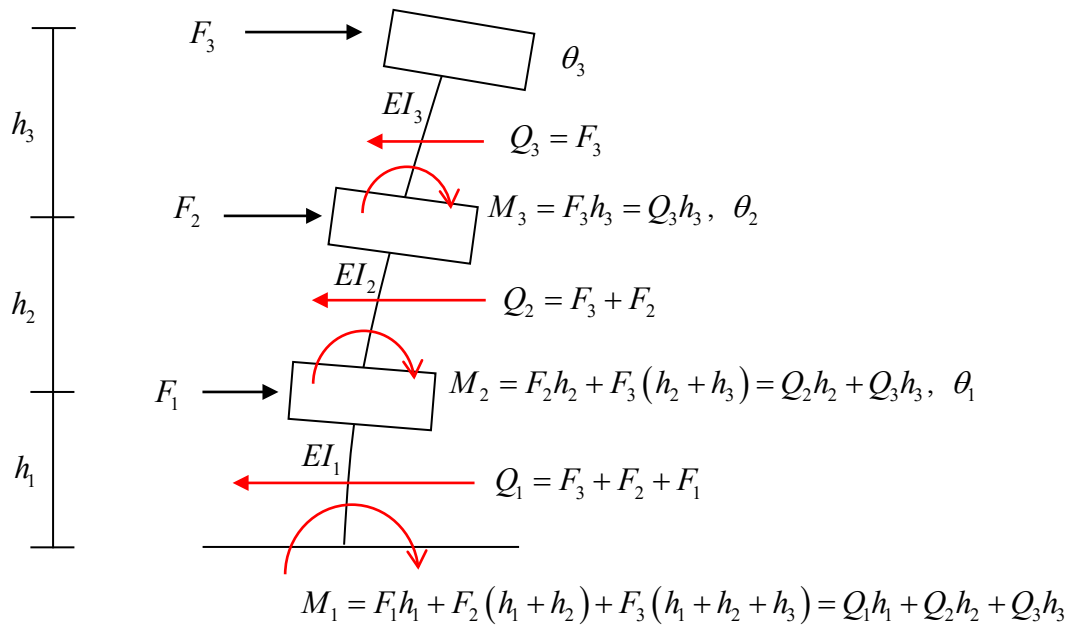


Figure 7-1-5 Moment and shear force of lumped mass model

In general

$$M_i = \sum_{j=i}^N Q_j h_j \quad (7-1-11)$$

Note that if we consider the sign of coordinate

$$M_{yi} = -\sum_{j=i}^N Q_{xj} h_j, \quad M_{xi} = \sum_{j=i}^N Q_{yj} h_j \quad (7-1-12)$$

From the beam theory

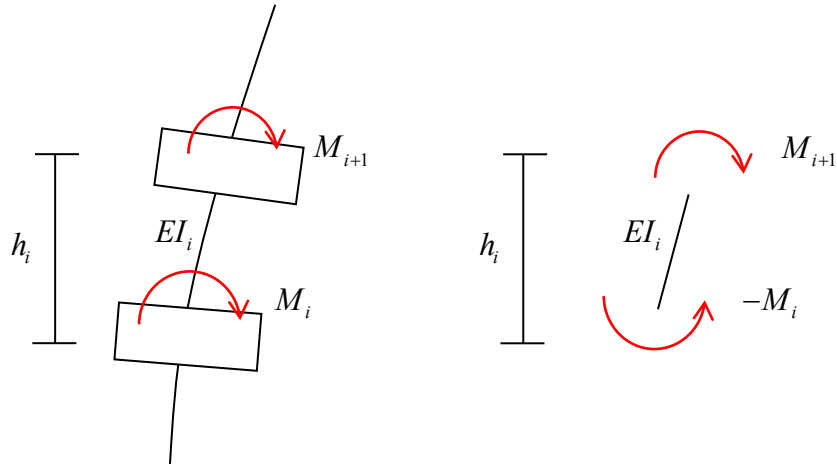
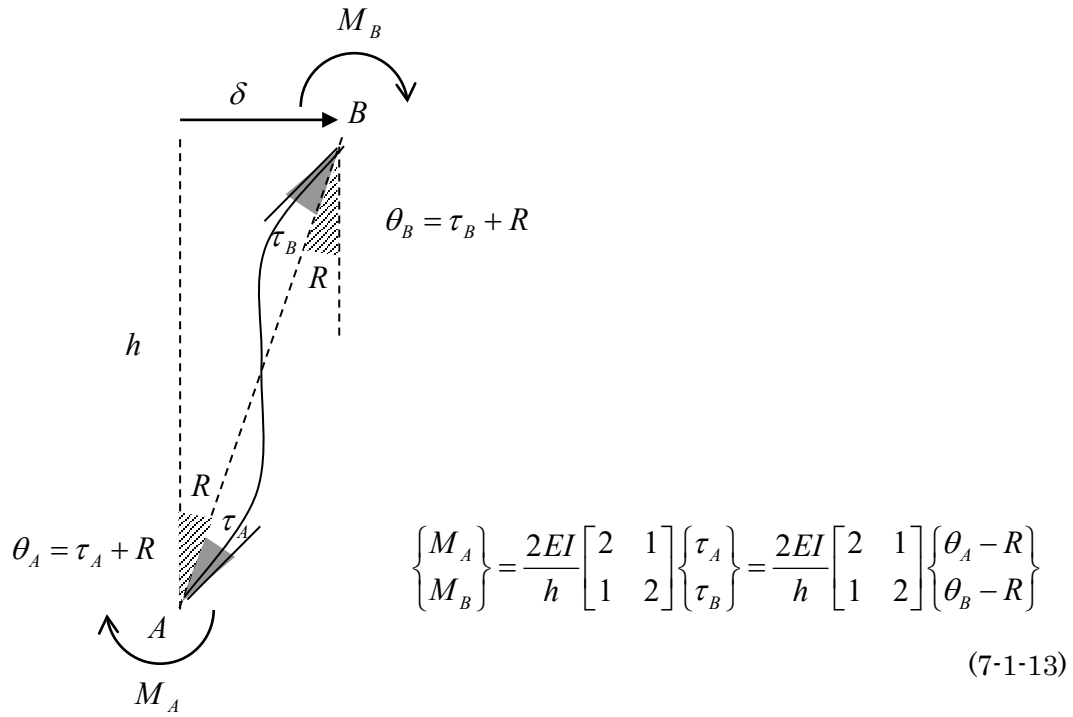


Figure 7-1-6 Moment and rotational deformation

Substituting

$$M_A = -M_i, \quad M_B = M_{i+1}, \quad \tau_A = \theta_{i-1}, \quad \tau_B = \theta_{i-1} + \Delta\theta_i, \quad EI = EI_i, \quad h = h_i$$

$$\begin{Bmatrix} -M_i \\ M_{i+1} \end{Bmatrix} = \frac{2EI_i}{h_i} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \theta_{i-1} \\ \theta_{i-1} + \Delta\theta_i \end{Bmatrix} = \frac{2EI_i}{h_i} \begin{Bmatrix} 3\theta_{i-1} + \Delta\theta_i \\ 3\theta_{i-1} + 2\Delta\theta_i \end{Bmatrix} \quad (7-1-14)$$

Therefore, the equivalent flexural stiffness can be obtained as

$$EI_i = \frac{h_i}{2\Delta\theta_i} (M_{i+1} + M_i), \quad i = 1, \dots, n \quad (7-1-15)$$

$$EI_{n+1} = \frac{h_{n+1}}{2\Delta\theta_{n+1}} (M_{n+1})$$

The increment of rotational deformation  $\Delta\theta_i$  is the difference of floor angle. Therefore,

$$\begin{aligned}\Delta\theta_1 &= \theta_1 \\ \Delta\theta_i &= \theta_i - \theta_{i-1}, \quad i = 2, \dots, n+1\end{aligned}\tag{7-1-16}$$

From the beam theory, the flexural deformation is

$$\delta = \frac{h^3}{12EI}Q + \frac{h}{2}(\theta_A + \theta_B)\tag{7-1-17}$$

$$\therefore Q = -\frac{M_A + M_B}{h} = -\frac{6EI}{h^2}\{(\theta_A + \theta_B) - 2R\}, \quad R = \frac{\delta}{h}$$

Therefore, the flexural deformation of i-th story is obtained as

$$\begin{aligned}D_{F1} &= \frac{h_1^3}{12EI_1}Q_1 + \frac{h_1}{2}\theta_1 \\ D_{Fi} &= \frac{h_i^3}{12EI_i}Q_i + \frac{h_i}{2}(\theta_{i-1} + \theta_i), \quad i = 2, \dots, n+1\end{aligned}\tag{7-1-18}$$

The shear deformation is then calculated substituting the flexural deformation from the story drift as

$$D_{Si} = D_i - D_{Fi}\tag{7-1-19}$$

Under the nonlinear push over analysis, it is generally assumed that the flexural component is elastic and only the shear component is considered as nonlinear.

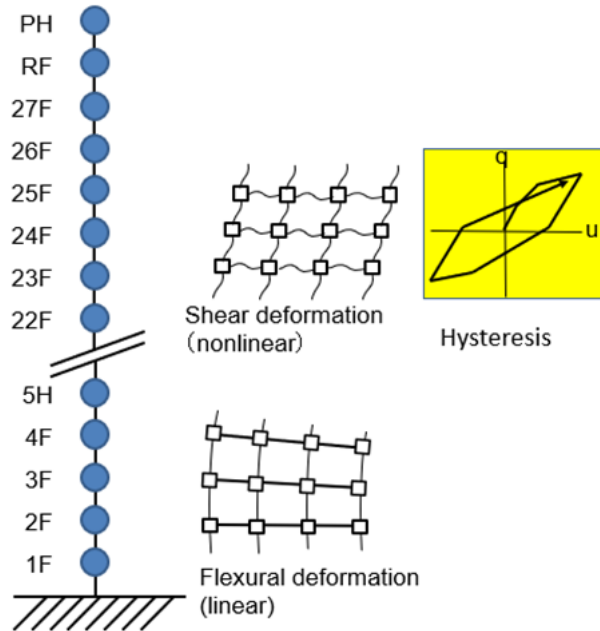


Figure 7-1-7 Decomposition of shear and flexural deformation

Therefore, the lumped mass model is obtained from the following procedure:

In the first step of push-over analysis (in elastic stage)

- 1) Calculate equivalent plane ( $z = ax + by + c$ ) of each floor to obtain the flexural angles  $a_i$  or  $b_i$
- 2) Calculate increment of angle  $\Delta\theta_i = a_i - a_{i-1}$  or  $\Delta\theta_i = b_i - b_{i-1}$

- 3) Calculate the flexural stiffness

$$EI_i = \frac{h_i}{2\Delta\theta_i} (M_{i+1} + M_i) \quad (7-1-20)$$

- 4) Calculate the flexural deformation

$$D_{xFi} = \frac{h_i^3}{12EI_i} Q_i - \frac{h_i}{2} (a_{i-1} + a_i) \quad \text{or} \quad D_{yFi} = \frac{h_i^3}{12EI_i} Q_i + \frac{h_i}{2} (b_{i-1} + b_i) \quad (7-1-22)$$

- 5) Calculate the shear deformation

$$D_{Si} = D_i - D_{Fi} \quad (7-1-23)$$

From the next step, we use the same flexural stiffness obtained previously.

- 6) Calculate increment of angle

$$\Delta\theta_i = \frac{h_i}{2EI_i} (M_{i+1} + M_i) \quad (7-1-24)$$

- 7) Calculate flexural angle of each floor

$$a_i = \sum_{k=1}^i \Delta\theta_k \quad \text{or} \quad b_i = \sum_{k=1}^i \Delta\theta_k \quad (7-1-25)$$

- 8) Calculate the flexural deformation

$$D_{xFi} = \frac{h_i^3}{12EI_i} Q_i - \frac{h_i}{2} (a_{i-1} + a_i) \quad \text{or} \quad D_{yFi} = \frac{h_i^3}{12EI_i} Q_i + \frac{h_i}{2} (b_{i-1} + b_i) \quad (7-1-26)$$

- 9) Calculate the shear deformation

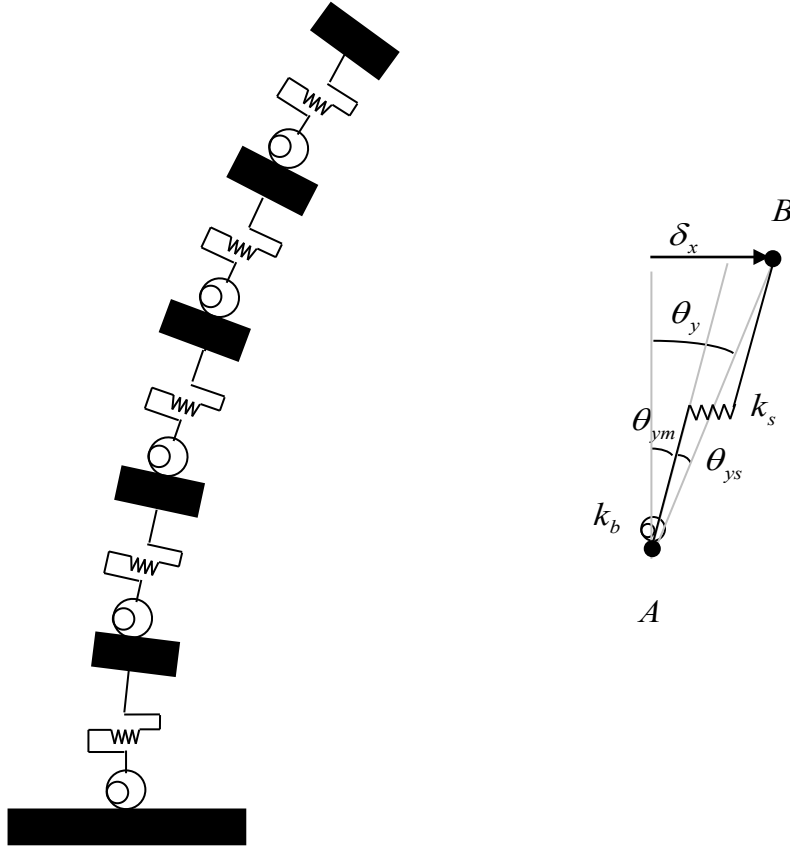
$$D_{Si} = D_i - D_{Fi} \quad (7-1-27)$$

- 10) The relationship between the shear deformation and the shear force is idealized as a nonlinear hysteresis model of the shear spring of each story.

## b) Nonlinear flexural model

To consider nonlinear flexural component, the model to separate shear deformation and bending eformation is used.

Reference) Akira Wada, et. Al. “Response Control Design of Buildings”, Maruzen (in Japanese), 1998



$$\theta_y = \theta_{ym} + \theta_{ys} = \frac{\delta_x}{h} \quad (7-1-28)$$

$$\begin{aligned} \downarrow \quad \theta_{ym} &= \frac{M_y}{k_b}, \quad \delta_{xs} = \frac{Q_x}{k_s}, \quad Q_x = \frac{M_y}{h} \\ \delta_x &= \theta_{ym}h + \theta_{ys}h = \frac{M_y}{k_b}h + \frac{Q_x}{k_s} = \frac{Q_x}{k_b}h^2 + \frac{Q_x}{k_s} = \left( \frac{h^2}{k_b} + \frac{1}{k_s} \right) Q_x \end{aligned} \quad (7-1-29)$$

Therefore, the relationship between the displacement and the force is expressed as follows:

$$Q_x = k_x \delta_x, \quad k_x = \frac{1}{\left( \frac{h^2}{k_b} + \frac{1}{k_s} \right)} \quad (7-1-30)$$

From nodal displacement,

$$\delta_x = u_{xB} - u_{xA} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \end{Bmatrix} \quad (7-1-31)$$

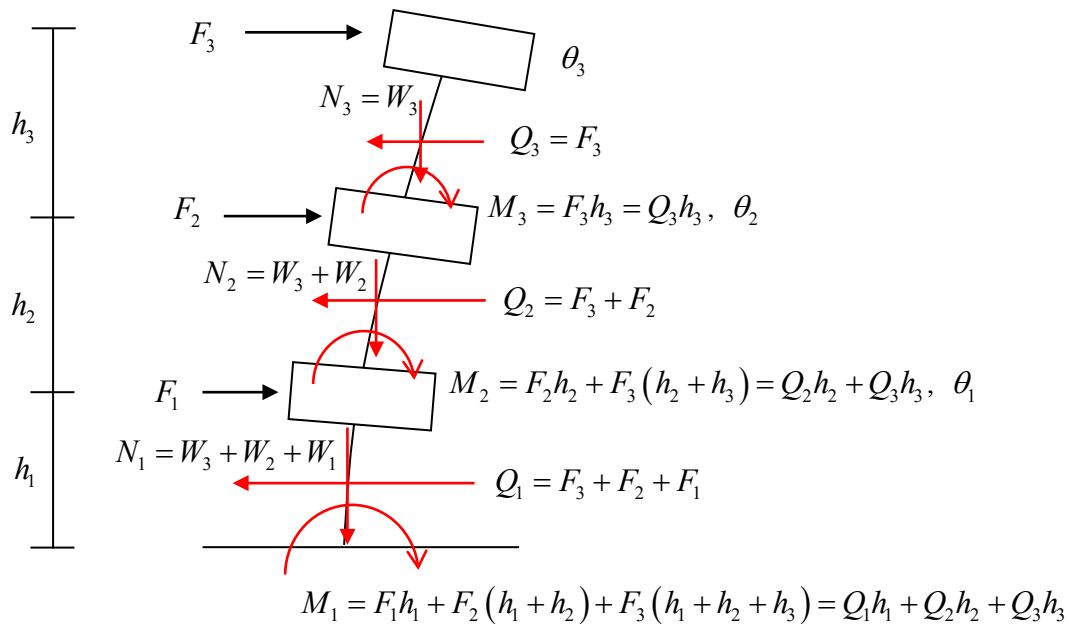


Figure 7-1-5 Moment and shear force of lumped mass model

By substituting

$$M_A = -M_i, \quad M_B = M_{i+1}, \quad Q_A = Q_i, \quad Q_B = Q_{i+1}, \quad N_A = N_i, \quad N_B = N_{i+1},$$

$$\theta_A = \theta_{i-1}, \quad \theta_B = \theta_i, \quad u_A = u_{i-1}, \quad u_B = u_i, \quad \delta_A = \delta_i, \quad \delta_B = \delta_{i+1}, \quad h = h_i$$

the lumped mass model is obtained from the following procedure from the push-over analysis.

- 1) Calculate equivalent plane ( $z = ax + by + c$ ) of each floor to obtain the flexural angles  $a_i$  or  $b_i$  and the vertical location at the center of gravity  $(x_{ci}, y_{ci})$  as  $z_{ci} = ax_{ci} + by_{ci} + c$ .

- 2) Calculate shear deformation

$$\delta_{s1} = u_1 - \eta h_1 \theta_1 \quad (7-1-32)$$

$$\delta_{si} = u_i - u_{i-1} - \eta h_i \theta_i, \quad i = 2, \dots, n+1 \quad (7-1-33)$$

Note that  $\eta = -1$  for  $\theta_i = a_i$  (x-direction) and  $\eta = 1$  for  $\theta_i = b_i$  (y-direction)

- 3) Calculate the shear stiffness

$$k_{si} = Q_{si} / \delta_{si} \quad (7-1-34)$$

- 4) Calculate axial deformation

$$\varepsilon_{n1} = \delta_1 \quad (7-1-35)$$

$$\varepsilon_{ni} = \delta_i - \delta_{i-1}, \quad i = 2, \dots, n+1 \quad (7-1-36)$$

Note that  $\delta_i = z_{ic} - \sum_{k=1}^i h_k$

- 5) Calculate the axial stiffness

$$k_{ni} = N_i / \varepsilon_{ni} \quad (7-1-37)$$

6) Calculate moment at each floor

$$M_i = \eta \sum_{j=i}^N Q_j h_j \quad (7-1-38)$$

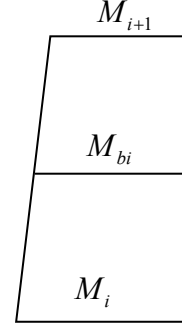
Note that  $\eta = -1$  for (x-direction) and  $\eta = 1$  for (y-direction)

7) Moment of the bending spring is

$$M_{bi} = M_i, \quad i = 1, \dots, n \quad (7-1-39)$$

The rotational deformation of the bending spring is

$$\phi_{ii} = \theta_i - \theta_{i-1} \quad (7-1-40)$$



The bending moment and the angle are transformed to the equivalent shear force and the equivalent story drift as follows:

$$\text{Equivalent shear force } Q_b = \frac{M_b}{h} \rightarrow Q_b = \frac{k_b}{h^2} \delta_b = k_{bs} \delta_b$$

$$\text{Equivalent story drift } \delta_b = h\phi$$

$$\text{Equivalent stiffness } Q_b = k_{bs} \delta_b, \quad k_{bs} = \frac{k_b}{h^2}$$

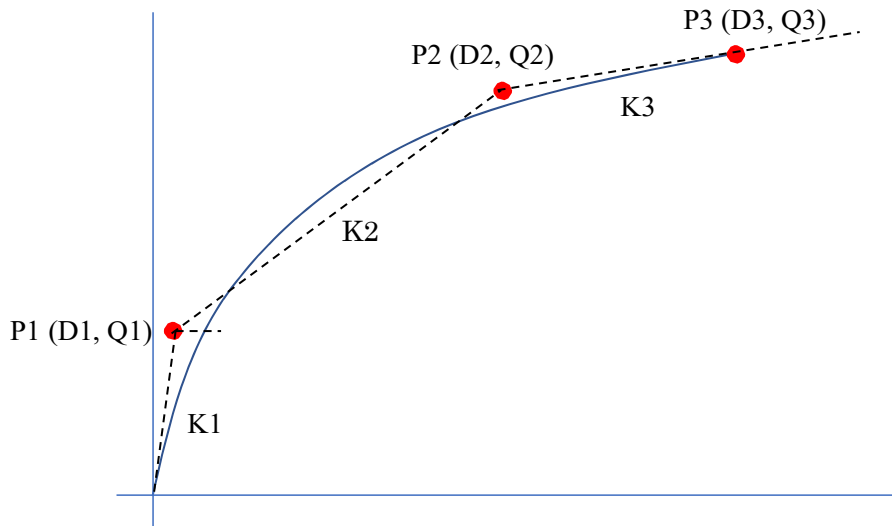
After finding the tri-linear model for  $Q_b - \delta_b$  relationship, it is returned to  $M_b - \phi$  relationship as,

$$M_b = Q_b h, \quad \phi = \frac{\delta_b}{h}, \quad k_b = k_{bs} h^2$$

In dynamic analysis, the rotational inertia at each floor is neglected.

### c) Trilinear modeling of push-over curve

From the push over results up to the ultimate deformation (for example, up to 1/50 drift ratio), the relationship between the story drift (shear  $\delta_s$ , bending  $\delta_b$ ) and the shear force (shear  $Q_s$ , bending  $Q_b$ ) of each story is transformed into a tri-linear skeleton.



#### < Case 1 >

When the drift ratio (drift divided by the story height) of the last point is less than the minimum value (for example, 1/1000)

The skeleton is assumed to be linear.

P1 (D1, Q1)

The last point is P1

$$K1 = Q1/D1$$

P2(D2, Q2)

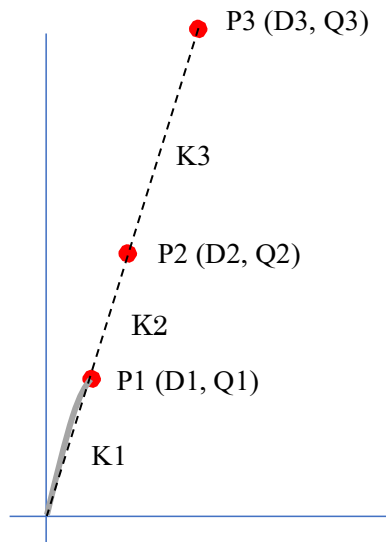
$$D2 = 2 \times D1$$

$$K2 = K1$$

P3(D3, Q3)

$$D3 = 4 \times D1$$

$$K3 = K1$$



**< Case 2 >**

When the last stiffness is large (for example, tangent stiffness  $> 0.1 K_1$  (initial stiffness))

$P_1(D_1, Q_1)$

Find initial stiffness  $K_1$

Find  $Q_1$  that is the force when the tangent stiffness becomes  $0.8K_1$  and determine  $D_1 = Q_1/K_1$

$P_2(D_2, Q_2)$

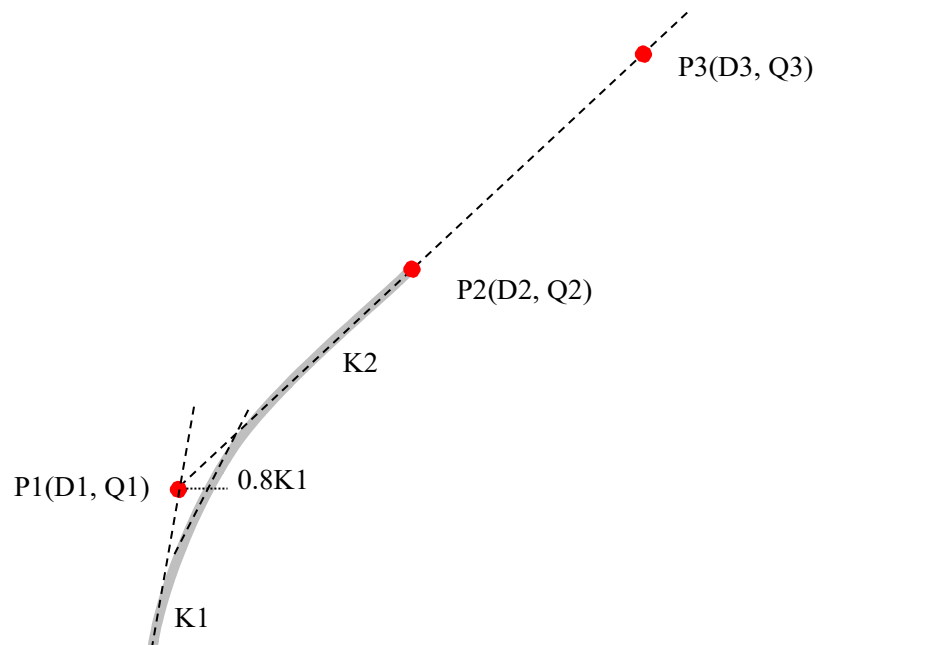
The last point is  $P_2$ .

$K_2$  is the stiffness between  $P_1$  and  $P_2$

$P_3(D_3, Q_3)$

$D_3 = 2 \times D_2$

$K_3 = K_2$



**< Case 3 >**

When the last stiffness is small (for example, tangent stiffness  $< 0.1 K_1$  (initial stiffness))

P1(D1, Q1)

Find initial stiffness  $K_1$

Find Q1 that is the force when the tangent stiffness becomes  $0.8K_1$  and determine  $D_1 = Q_1/K_1$

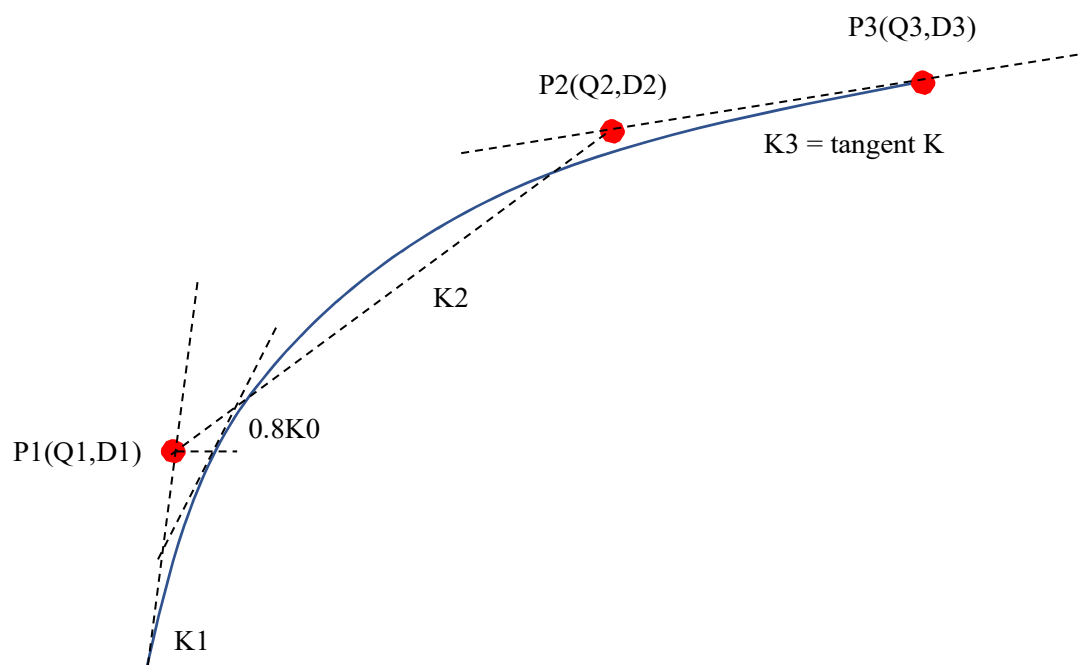
P2(D2, Q2)

P2 is decided to be the same energy between push-over analysis and the model up to P3.

P3(D3, Q3)

P3 is the last point of push-over analysis

$K_3$  is the tangent stiffness at P3



## 8. P-D effect

The formulation in this chapter is based on the following book:

James F. Doyle, “Static and Dynamic Analysis of Structures”, Kluwer Academic Publishers, 1991

### a) Equilibrium of the beam with an axial load

We consider equilibrium of the beam with a slight displacement with an axial load.

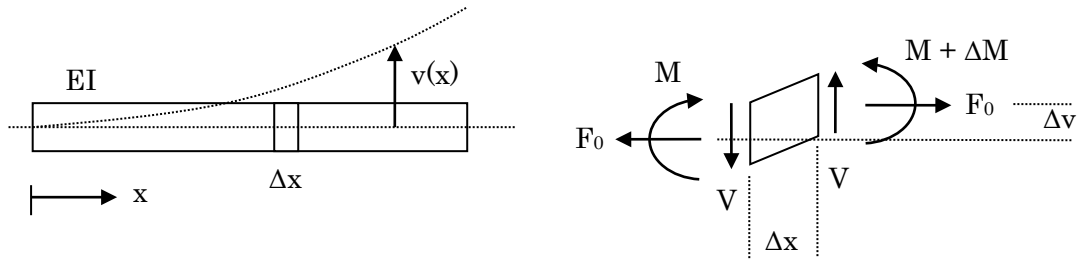


Figure 7-2-1 Equilibrium of small beam segment slightly deformed

Assuming small deflection, the balance of moment on the small segment “ $\Delta x$ ” gives

$$\Delta M + V(\Delta x) - F_0(\Delta v) = 0 \quad (8-1-1)$$

Therefore

$$\frac{dM}{dx} + V - F_0 \frac{dv}{dx} = 0 \quad (8-1-2)$$

From the relationship,  $M = EI \frac{d^2 v}{dx^2}$ , the governing differential equation for the deflection shape is

$$EI \frac{d^4 v}{dx^4} - F_0 \frac{d^2 v}{dx^2} = 0 \quad (8-1-3)$$

The general solutions are,

for compression loading ( $F_0 < 0$ ):

$$v(x) = c_1 \cos kx + c_2 \sin kx + c_3 x + c_4, \quad k^2 = -F_0 / EI, \quad (8-1-4)$$

for tensile loading ( $F_0 > 0$ ):

$$v(x) = c_1 \cosh kx + c_2 \sinh kx + c_3 x + c_4, \quad k^2 = F_0 / EI \quad (8-1-5)$$

### b) Geometric stiffness matrix of the beam with an axial load

We assume that the axial force is constant and compressive. From the general solution, Eq. (8-1-4), at  $x = 0$

$$v(0) = v_1 = c_1 + c_4, \quad \frac{dv(0)}{dx} = \phi_1 = kc_2 + c_3 \quad (8-1-6)$$

Consequently, the deflected shape is

$$v(x) = c_1(\cos kx - 1) + c_2(\sin kx - kx) + v_1 + \phi_1 x \quad (8-1-7)$$

Similarly at the end of other node,

$$v(L) = v_2 = c_1(\cos kL - 1) + c_2(\sin kL - kL) + v_1 + \phi_1 L \quad (8-1-8)$$

$$\frac{dv(L)}{dx} = \phi_2 = -kc_1 \sin kL + kc_2 \cos kL + \phi_1 \quad (8-1-9)$$

Then, the coefficients,  $c_1, c_2$ , can be arranged as,

$$\begin{bmatrix} (1-C) & (\xi-S) \\ \xi S & \xi(1-C) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} v_1 + \phi_1 L - v_2 \\ \phi_1 L - \phi_2 L \end{bmatrix} \quad (8-1-10)$$

where,

$$C = \cos kL, \quad S = \sin kL, \quad \xi = kL \quad (8-1-11)$$

Solving this equation by Cramer's rule gives

$$c_1 = [v_1 \xi(1-C) + \phi_1 L(S - \xi C) - v_2 \xi(1-C) + \phi_2 L(\xi - S)] / \Delta \quad (8-1-12)$$

$$c_2 = [-v_1 \xi S + \phi_1 L(1-C - \xi S) + v_2 \xi S + \phi_2 L(C - 1)] / \Delta \quad (8-1-13)$$

where

$$\Delta = \xi(2 - 2C - \xi S) \quad (8-1-14)$$

Now we can rewrite the deflection function in terms of the nodal degrees of freedom. The moment and shear force distributions can be obtained as

$$M(x) = EI \frac{d^2 v}{dx^2} = EI [-k^2 c_1 \cos kx - k^2 c_2 \sin kx] \quad (8-1-15)$$

$$V(x) = -EI \frac{d^3 v}{dx^3} + F_0 \frac{dv}{dx} = -EI k^2 [\phi_1 - kc_2] \quad (8-1-16)$$

Calculating nodal loads,  $V(0) = -V_1$ ,  $M(0) = -M_1$ ,  $V(L) = V_1$ ,  $M(L) = M_1$ , the stiffness matrix is

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \frac{EI}{L^3} \frac{\xi^2}{\Delta} \begin{bmatrix} \xi^2 S & \xi L(1-C) & -\xi^2 S & \xi L(1-C) \\ -L^2(\xi C - S) & -\xi L(1-C) & L^2(\xi - S) & \\ \xi^2 S & -\xi L(1-C) & \\ sym. & -L^2(\xi C - S) \end{bmatrix} \begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{bmatrix} \quad (8-1-17)$$

### c) Approximation of geometric stiffness matrix

We simplify the geometric stiffness matrix to be linear in the loading  $F_0$ .

Using the series expansion for the sine and cosine terms, the determinant is,

$$\begin{aligned} \Delta &= \xi(2 - 2C - \xi S) \\ &\approx \xi \left[ 2 - 2 \left( 1 - \xi^2/2 + \xi^4/24 - \xi^6/720 + \dots \right) C - \xi \left( \xi - \xi^3/6 + \xi^5/120 - \dots \right) \right] \\ &\approx \xi^5 \left[ 1 - \xi^2/15 + \dots \right] / 12 \end{aligned} \quad (8-1-18)$$

also

$$\frac{1}{\Delta} = \frac{12}{\xi^5} \left[ 1 + \xi^2/15 + \dots \right] \quad (8-1-19)$$

We now do the expansion on the stiffness terms. For example,

$$k_{11} = \frac{EI}{L^3} \frac{\xi^2}{\Delta} (\xi^2 S) = \frac{EI}{L^3} \left[ \xi^4 \left( \xi - \xi^3/6 + \dots \right) \right] \frac{12}{\xi^5} \left[ 1 + \xi^2/15 + \dots \right] = \frac{EI}{L^3} 12 \left[ 1 - \xi^2/10 + \dots \right] \quad (8-1-20)$$

Substituting  $\xi^2 = k^2 L^2 = -F_0 L / EI$ ,

$$k_{11} = \frac{EI}{L^3} [12] + \frac{F_0}{L} \left[ \frac{12}{10} \right] \quad (8-1-21)$$

In the same manner, we can expand for all the stiffness terms to get the stiffness matrix as

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ sym. & & & 4L^2 \end{bmatrix} + \frac{F_0}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ & 4L^2 & -3L & -L^2 \\ & & 36 & -3L \\ sym. & & & 4L^2 \end{bmatrix} \quad (8-1-22)$$

We can write as

$$[k] = [k_E] + [k_G] \quad (8-1-23)$$

where,  $[k_E]$ : the element elastic stiffness,  $[k_G]$ : the element geometric stiffness

d) Implementation for beam element

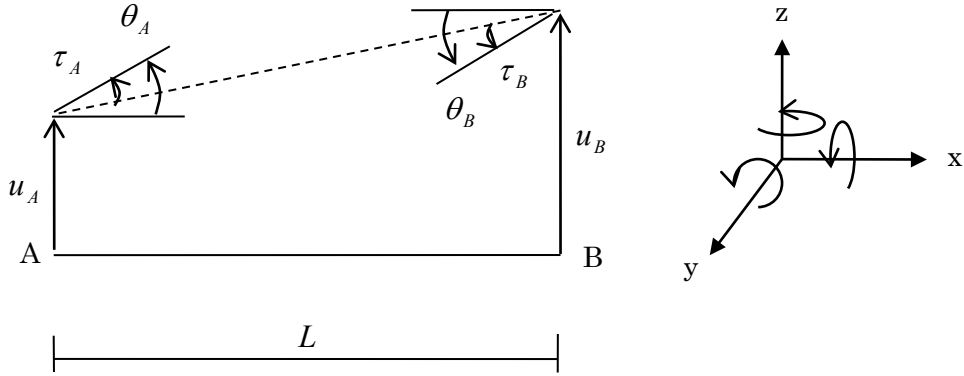


Figure 7-2-2 Including node movement

For beam element,

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \frac{2EI}{L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \tau_A \\ \tau_B \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} \begin{bmatrix} \tau_A \\ \tau_B \end{bmatrix} \quad (8-1-24)$$

Including node movement,

$$\begin{bmatrix} \tau_A \\ \tau_B \end{bmatrix} = \begin{bmatrix} \frac{1}{L} & 1 & -\frac{1}{L} & 0 \\ \frac{1}{L} & 0 & -\frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_A \\ \theta_A \\ u_B \\ \theta_B \end{bmatrix} \quad (8-1-25)$$

$$\begin{aligned} \begin{bmatrix} Q_A \\ M_A \\ Q_B \\ M_B \end{bmatrix} &= \frac{EI}{L^3} \begin{bmatrix} \frac{1}{L} & \frac{1}{L} \\ 1 & 0 \\ -\frac{1}{L} & -\frac{1}{L} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} \begin{bmatrix} \frac{1}{L} & 1 & -\frac{1}{L} & 0 \\ \frac{1}{L} & 0 & -\frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_A \\ \theta_A \\ u_B \\ \theta_B \end{bmatrix} \\ &= \frac{EI}{L^3} \begin{bmatrix} 6L & 6L \\ 4L^2 & 2L^2 \\ -6L & -6L \\ 2L^2 & 4L^2 \end{bmatrix} \begin{bmatrix} \frac{1}{L} & 1 & -\frac{1}{L} & 0 \\ \frac{1}{L} & 0 & -\frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_A \\ \theta_A \\ u_B \\ \theta_B \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ sym. & & & 4L^2 \end{bmatrix} \begin{bmatrix} u_A \\ \theta_A \\ u_B \\ \theta_B \end{bmatrix} \end{aligned}$$

From (8-1-22), the geometric stiffness matrix will be

$$[k_G] = \frac{F_0}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ & 4L^2 & -3L & -L^2 \\ & & 36 & -3L \\ sym. & & & 4L^2 \end{bmatrix} \quad (8-1-26)$$

Therefore, the stiffness equation will be

$$\begin{bmatrix} Q_A \\ M_A \\ Q_B \\ M_B \end{bmatrix} = \left( \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ sym. & & & 4L^2 \end{bmatrix} + \frac{F_0}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ & 4L^2 & -3L & -L^2 \\ & & 36 & -3L \\ sym. & & & 4L^2 \end{bmatrix} \right) \begin{bmatrix} u_A \\ \theta_A \\ u_B \\ \theta_B \end{bmatrix}$$

e) Implementation for column element

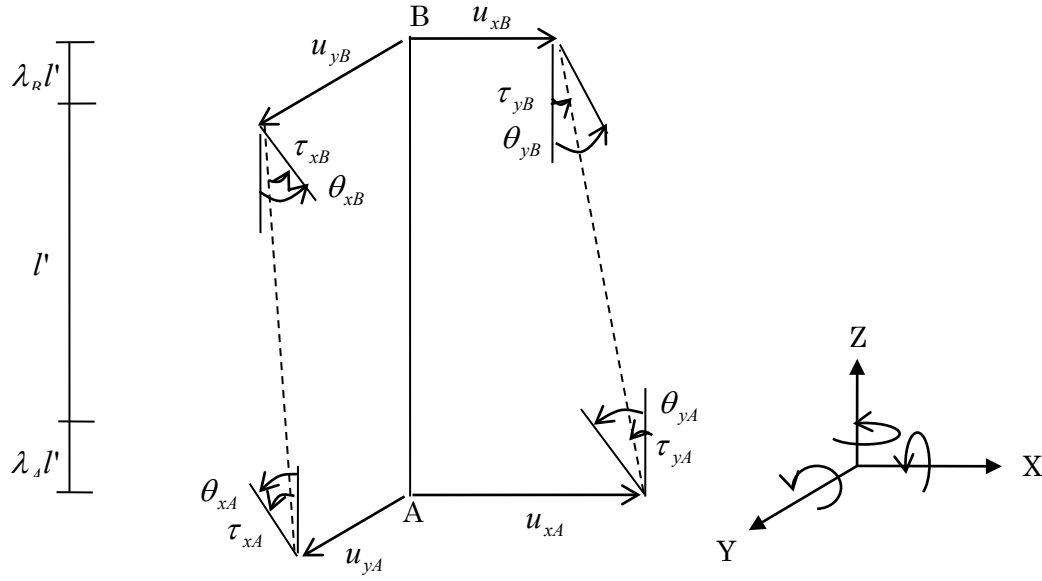


Figure 7-2-3 Including node movement

$$\begin{bmatrix} M_{yA} \\ M_{yB} \end{bmatrix} = \frac{2EI}{L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \tau_{yA} \\ \tau_{yB} \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} \begin{bmatrix} \tau_{yA} \\ \tau_{yB} \end{bmatrix} \quad \text{in X-Z plane} \quad (8-1-27)$$

$$\begin{bmatrix} M_{xA} \\ M_{xB} \end{bmatrix} = \frac{2EI}{L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \tau_{xA} \\ \tau_{xB} \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} \begin{bmatrix} \tau_{xA} \\ \tau_{xB} \end{bmatrix} \quad \text{in Y-Z plane} \quad (8-1-28)$$

Including node movement,

$$\begin{bmatrix} \tau_{yA} \\ \tau_{yB} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & 1 & \frac{1}{L} & 0 \\ -\frac{1}{L} & 0 & \frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_{xA} \\ \theta_{yA} \\ u_{xB} \\ \theta_{yB} \end{bmatrix} \quad \text{in X-Z plane} \quad (8-1-29)$$

$$\begin{bmatrix} \tau_{xA} \\ \tau_{xB} \end{bmatrix} = \begin{bmatrix} \frac{1}{L} & 1 & -\frac{1}{L} & 0 \\ \frac{1}{L} & 0 & -\frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_{yA} \\ \theta_{xA} \\ u_{yB} \\ \theta_{xB} \end{bmatrix} \quad \text{in Y-Z plane} \quad (8-1-30)$$

Note that the matrix for node movement in X-Z plane is different from that of beam element. The force-deformation relationship in X-Z plane is then,

$$\begin{aligned}
\begin{bmatrix} Q_{xA} \\ M_{yA} \\ Q_{xB} \\ M_{yB} \end{bmatrix} &= \frac{EI}{L^3} \begin{bmatrix} -\frac{1}{L} & -\frac{1}{L} \\ 1 & 0 \\ \frac{1}{L} & \frac{1}{L} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & 1 & \frac{1}{L} & 0 \\ \frac{1}{L} & 0 & \frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_{xA} \\ \theta_{yA} \\ u_{xB} \\ \theta_{yB} \end{bmatrix} \\
&= \frac{EI}{L^3} \begin{bmatrix} -6L & -6L \\ 4L^2 & 2L^2 \\ 6L & 6L \\ 2L^2 & 4L^2 \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & 1 & \frac{1}{L} & 0 \\ \frac{1}{L} & 0 & \frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_{xA} \\ \theta_{yA} \\ u_{xB} \\ \theta_{yB} \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ & 4L^2 & 6L & 2L^2 \\ & & 12 & 6L \\ sym. & & & 4L^2 \end{bmatrix} \begin{bmatrix} u_{xA} \\ \theta_{yA} \\ u_{xB} \\ \theta_{yB} \end{bmatrix}
\end{aligned}
\tag{8-1-31}$$

Considering the difference of sign of stiffness matrix in X-Z plane, the geometric stiffness matrix will be

$$[k_{xG}] = \frac{F_0}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ & 4L^2 & 3L & -L^2 \\ & & 36 & 3L \\ sym. & & & 4L^2 \end{bmatrix} \text{ in X-Z plane} \tag{8-1-32}$$

$$[k_{yG}] = \frac{F_0}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ & 4L^2 & -3L & -L^2 \\ & & 36 & -3L \\ sym. & & & 4L^2 \end{bmatrix} \text{ in Y-Z plane} \tag{8-1-33}$$

Therefore, changing the order of vector component, the force-deformation relationship of column will be

$$\begin{Bmatrix} Q_{xA} \\ Q_{xB} \\ M_{yA} \\ M_{yB} \\ Q_{yA} \\ Q_{yB} \\ M_{xA} \\ M_{xB} \\ N_{zA} \\ N_{zB} \\ M_{zA} \\ M_{zB} \end{Bmatrix} = [K] \begin{Bmatrix} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \\ u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix} + \frac{F_0}{30L} \begin{bmatrix} 36 & -36 & -3L & -3L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & 36 & 3L & 3L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3L & 3L & 4L^2 & -L^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3L & 3L & -L^2 & 4L^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 36 & -36 & 3L & 3L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -36 & 36 & -3L & -3L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3L & -3L & 4L^2 & -L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3L & -3L & -L^2 & 4L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \\ u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{Bmatrix}$$

$$= \left[ [K] + [K_G] \right] \left\{ \begin{array}{c} u_{xA} \\ u_{xB} \\ \theta_{yA} \\ \theta_{yB} \\ u_{yA} \\ u_{yB} \\ \theta_{xA} \\ \theta_{xB} \\ \delta_{zA} \\ \delta_{zB} \\ \theta_{zA} \\ \theta_{zB} \end{array} \right\} \quad (8-1-34)$$

where,

$$[K_G] = \frac{F_0}{30L} \left[ \begin{array}{cccccccccccc} 36 & -36 & -3L & -3L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & 36 & 3L & 3L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3L & 3L & 4L^2 & -L^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3L & 3L & -L^2 & 4L^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 36 & -36 & 3L & 3L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -36 & 36 & -3L & -3L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3L & -3L & 4L^2 & -L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3L & -3L & -L^2 & 4L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad (8-1-35)$$

Then, applying translation of Equation (2-2-17), the constitutive equation of the column is;

$$\left\{ \begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \end{array} \right\} = [K_C] \left\{ \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \right\} \quad (8-1-36)$$

where,

$$[K_C] = [T_C]^T [k_C] [T_C] + [T_{iC}]^T [K_G] [T_{iC}] \quad (8-1-37)$$

## 9. Unbalance force correction

### a) Procedure to correct unbalance force

In nonlinear analysis, sudden change of spring stiffness sometimes causes severe error for estimating element force. For example, estimation of spring force  $f_{i+1}$  is overestimated in Figure 9-1-1 and “unbalance force” is defined as,

$$\Delta f = f_{i+1} - f'_{i+1} \quad (9-1-1)$$

where,  $f'_{i+1}$  is the force on the nonlinear skeleton curve

The most preferable way to minimize the error is to adopt iterative calculations such as Newton-Raphson method. However, this iteration may consume calculation time significantly. Therefore, the following simple way is adopted to correct unbalance force:

- 1) Calculate unbalance displacement  $\Delta d$  from the unbalance force  $\Delta f$

$$\Delta d = \Delta f / k \quad (9-1-2)$$

where,  $k$  is the spring stiffness

- 2) Subtract unbalance displacement  $\Delta d$  from the increment displacement in the next step calculation

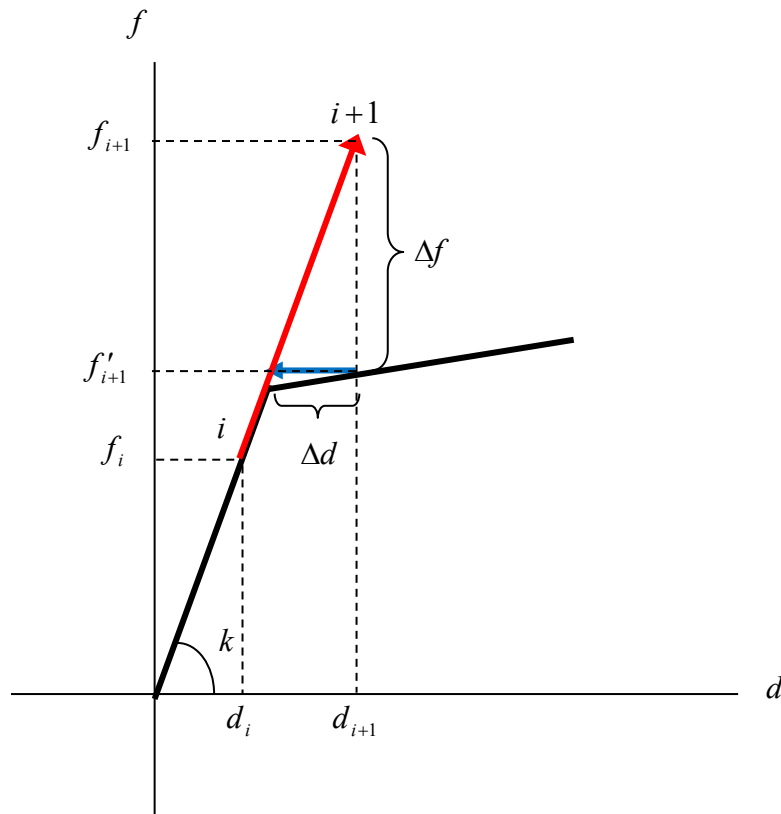


Figure 9-1-1 Unbalance force

### b) Unbalance force correction of MS model

For the Multi-spring model (MS model) of Column element, the sum of the unbalance forces of nonlinear vertical springs in the member section is calculated as:

$$\Delta N = \sum_{i=1}^5 \Delta f_i = \sum_{i=1}^5 (\Delta f_{c,i} + \Delta f_{s,i}) \quad (9-1-3)$$

where  $\Delta f_{c,i}$  : unbalance force of concrete spring,

$\Delta f_{s,i}$  : unbalance force of steel spring

The unbalance displacement is then calculated as:

$$\Delta D = \Delta N / \sum_{i=1}^5 k_i = \Delta N / \sum_{i=1}^5 (k_{c,i} + k_{s,i}) \quad (9-1-4)$$

where  $k_{c,i}$  : stiffness of concrete spring,

$k_{s,i}$  : stiffness of steel spring

In the next step calculation, the increment displacement of each spring is adjusted as follows:

$$\Delta d'_i = \Delta d_i - \Delta D \quad (9-1-5)$$

where  $\Delta d_i$  : increment displacement of i-th spring

$\Delta d'_i$  : adjusted increment displacement of i-th spring

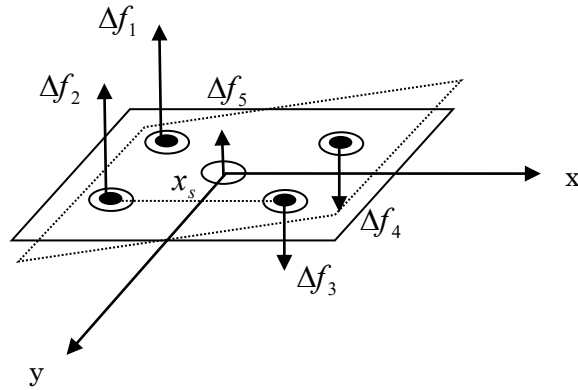


Figure 9-1-2 Unbalance force in MS-model

The same procedure is adopted for the MS model of Wall element.

## 10. Calculation of ground displacement

In STERA\_3D, the ground displacement is calculated from the ground acceleration data using FFT method and filtering techniques based on the description in the following reference:

**Reference:** Yorihiro Osaki, “Introduction of Spectral Analysis of Earthquake Ground Motion”, Kajima publishing corporation, 1981 (in Japanese)

### a) Discrete Fourier Transform

Assume that the acceleration data is collected at an interval,  $\Delta t = T / N$  and consists of the  $N$  measurement data  $x_m$  ( $m = 0, 1, 2, \dots, N-1$ ), where  $T$  is the period of the data that corresponds to the duration time of data. The coefficient of a Fourier series is obtained as:

$$C_k = \frac{1}{N} \sum_{m=0}^{N-1} x_m e^{-i(2\pi km / N)} \quad k = 0, 1, 2, \dots, N-1 \quad (10-1-1)$$

The inverse discrete Fourier transform is

$$x_m = \sum_{k=0}^{N-1} C_k e^{i(2\pi km / N)} \quad m = 0, 1, 2, \dots, N-1 \quad (10-1-2)$$

### b) Integration of the data in time domain

Assume  $y_m$  ( $m = 0, 1, 2, \dots, N-1$ ) is the integration of the discrete data  $x_m$  in time domain.

The data  $y_m$  is obtained by the following inverse discrete Fourier transform:

$$y_m = \left( \int_0^{t=m\Delta t} x_m dt \right) = \frac{N\Delta t}{2\pi} \sum_{k=0}^{N-1} S_k e^{i(2\pi km / N)} \quad m = 0, 1, 2, \dots, N-1 \quad (10-1-3)$$

where, the coefficients  $S_k$  are obtained from the coefficients  $C_k$  as,

$$S_0 = \frac{2\pi v_0}{N\Delta t} - 2 \sum_{k=0}^{N/2-1} \frac{\text{Im}(C_k)}{k} + \frac{\pi(N-1)C_0}{N}$$

$$S_k = \frac{\pi C_0}{N} [-1 + i \cos(\pi k / N)] - \frac{i C_k}{k}, \quad S_{N-k} = S_{N-k}^* \quad k = 1, 2, \dots, N/2-1 \quad (10-1-4)$$

$$S_{N/2} = -\frac{\pi C_0}{N}$$

The following band pass filter (Butterworth filter) in frequency domain is applied to the coefficient  $S_k$ .

$$G_B(f) = G_L(f)G_H(f) \quad (10-1-5)$$

$$G_L(f) = \sqrt{\frac{(f/f_L)^{2n}}{1+(f/f_L)^{2n}}} \quad (10-1-6)$$

$$G_H(f) = \sqrt{\frac{1}{1+(f/f_H)^{2n}}} \quad (10-1-7)$$

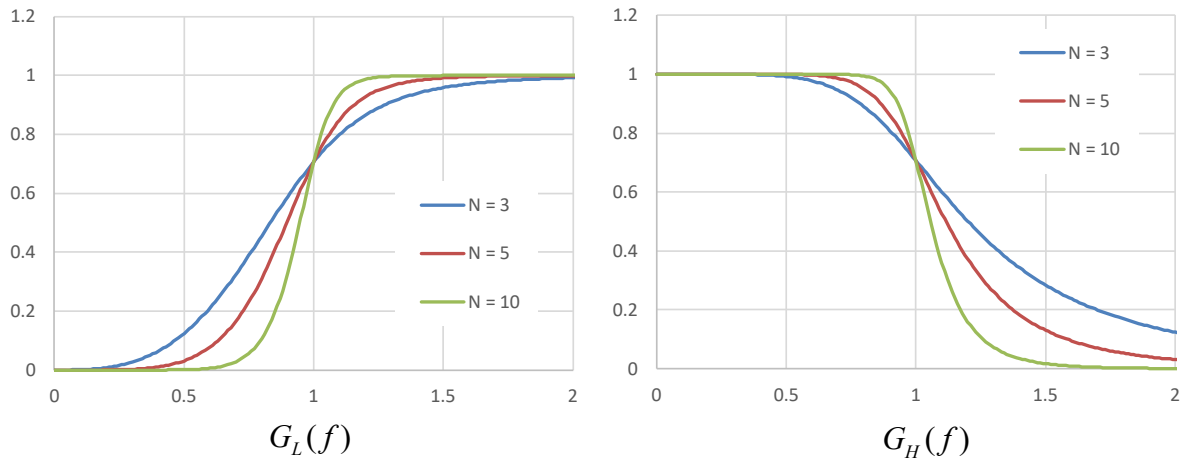


Figure 10-1-1 Butterworth filter

STERA\_3D adopts the following frequency parameters:

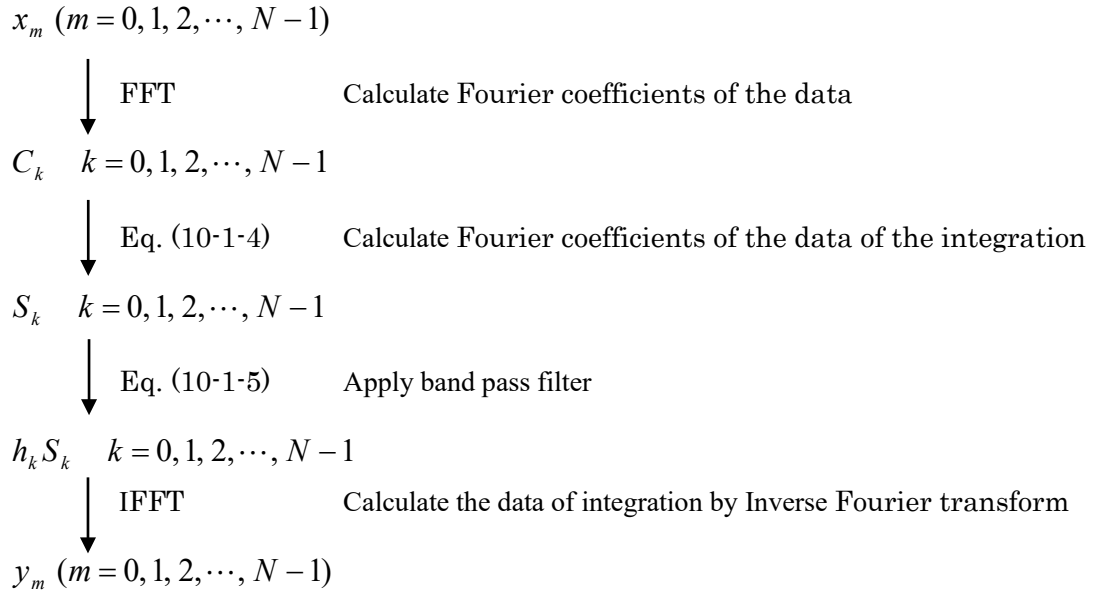
$$f_L = 0.1 \quad (\text{Hz})$$

$$f_H = 20 \quad (\text{Hz})$$

### c) Calculation flow

The ground acceleration data is integrated twice to obtain displacement data. Band pass filter is applied each time of the integration. The flow of calculation is summarized below:

[1] From acceleration data to velocity data



[2] From velocity data to displacement data

Repeat the above process again

## 11. Damage Index

### 11.1 Damage Index of RC Members

#### Reference:

- Young-Ji Park, A. H. Ang (1985) “Mechanistic Seismic Damage Model for Reinforced Concrete”, Journal of Structural Engineering, ASCE

#### 1) Park and Ang Damage Index

STERA\_3D adopts the following damage index, so called Park and Ang damage index, to evaluate the structural damage under earthquake.

$$D = \frac{\delta_m}{\delta_u} + \beta \frac{E_h}{Q_y \delta_u} = \frac{\mu_m}{\mu_u} + \beta \frac{E_h}{Q_y \delta_u} \quad (11-1-1)$$

where

- $\delta_m = \mu_m \delta_y$  : maximum deformation under an earthquake,
- $\delta_u = \mu_u \delta_y$  : ultimate deformation under a monotonic loading,
- $\mu_m$  : maximum ductility factor under an earthquake,
- $\mu_u$  : ultimate ductility factor under a monotonic loading,
- $\delta_y$  : yield deformation,
- $Q_y$  : yield strength,
- $\beta$  : parameter related to the cumulative loading effect,
- $E_h = \int dE$  : dissipated hysteretic energy.

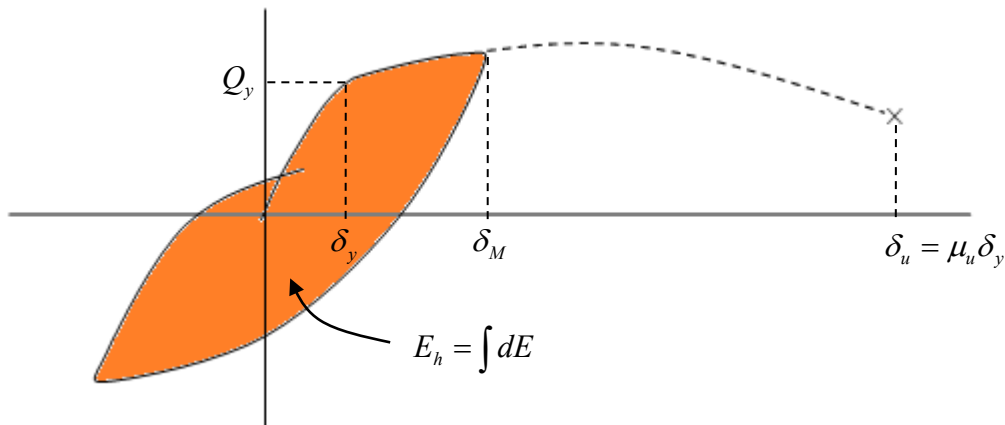


Figure 11-1-1 Force-deformation relationship of the member

The cumulative ductility factor is the ratio of the cumulative dissipated energy defined as

$$\eta = \frac{E_h}{Q_y \delta_y} \quad (11-1-2)$$

The damage index can be rewritten as

$$D = \frac{\mu_m}{\mu_u} + \beta \frac{\eta}{\mu_u} \quad (11-1-3)$$

## 2) RC Beam and Column

### Ultimate ductility factor

According to Park and Ang (1985), the ultimate ductility factor,  $\mu_u$ , for reinforced concrete beams and columns is highly variable and depends on the failure mode of the member as shown in Figure 11-2.

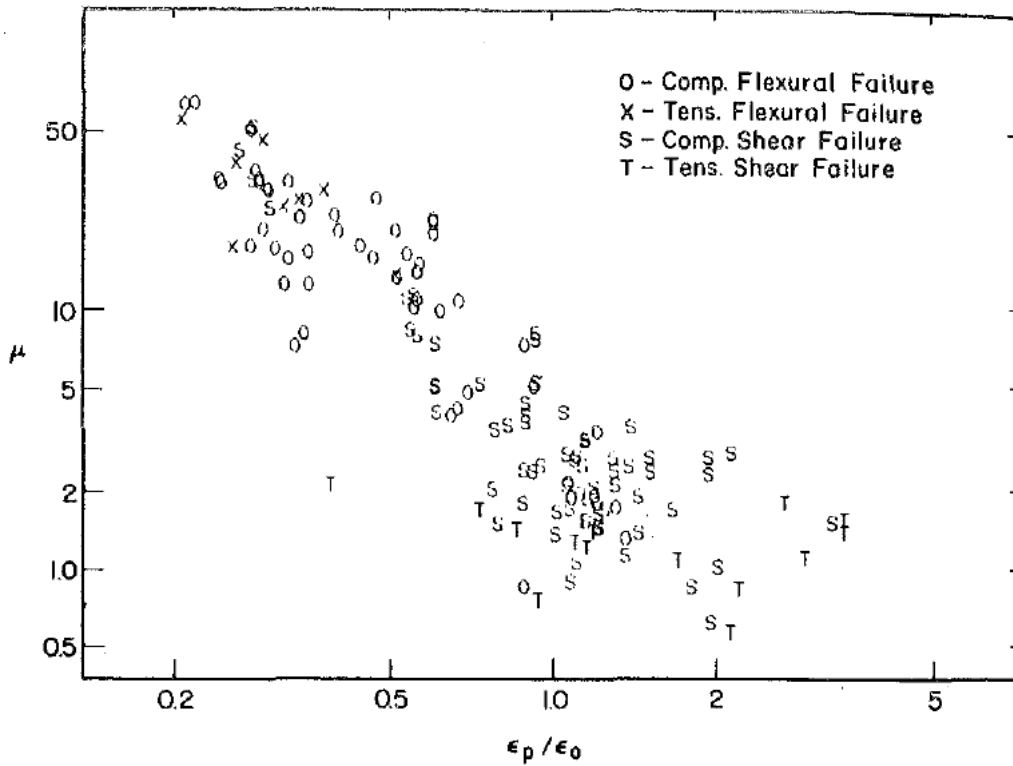


Figure 11-1-2 Ultimate ductility factor and failure mode (Park and Ang (1985))

In case of the flexural failure, the value is greater than 10. Therefore, in STERA\_3D,

$$\mu_u = 15$$

is adopted for the nonlinear flexural springs at both ends of the reinforced concrete beams and columns.

### Parameter $\beta$

The parameter  $\beta$  represents the effect of cyclic loading on damage. According to Park and Ang (1985),  $\beta$  is calculated as,

$$\beta = \left( -0.447 + 0.073 \frac{l}{d} + 0.24n_0 + 0.314p_t \right) \times 0.7^{\rho_w} \quad (11-1-4)$$

where

- $l/d$  : shear span ratio (replaced by 1.7 if  $l/d < 1.7$ ),
- $n_0$  : normalized axial stress (replaced by 0.2 if  $n_0 < 0.2$ ),
- $p_t$  : longitudinal steel ratio as a percentage (replaced by 0.75% if  $p_t < 0.75\%$ ),
- $\rho_w$  : confinement ratio.

Figure 11-1-3 shows the comparison between the calculated and experimental results of  $\beta$ . The applicable range of the above equation is

$$1.0 < l/d < 6.6$$

$$0 \leq n_0 < 0.52$$

$$0.2 < \rho_w < 2.0$$

$$15.9 \text{ MPa} < f'_c \text{ (concrete strength)} < 41.4 \text{ MPa} \quad (11-1-5)$$

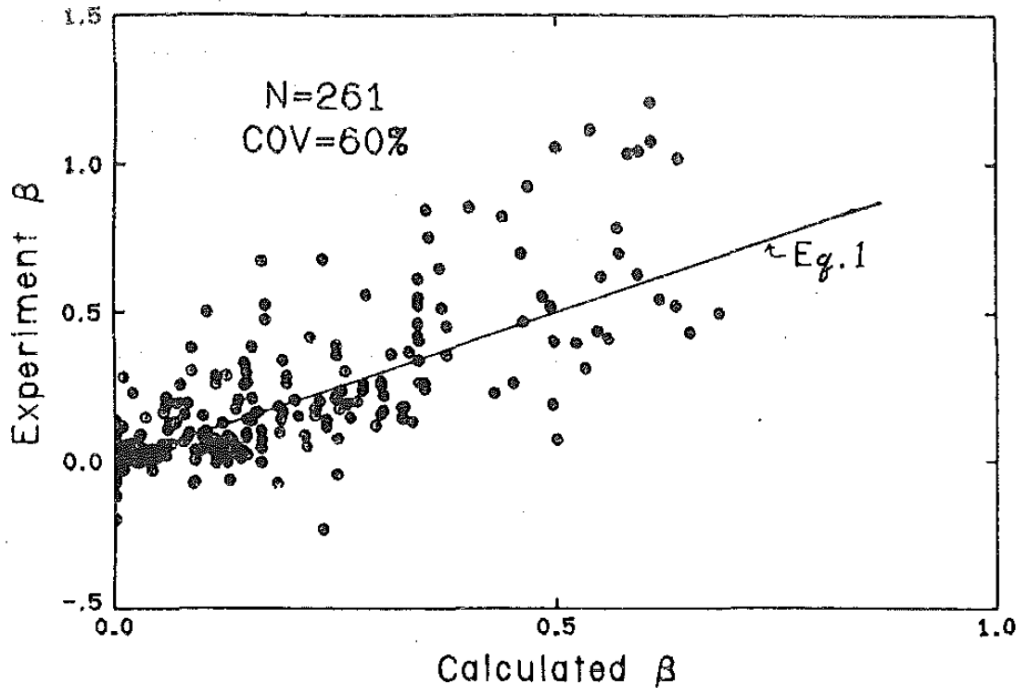


Figure 11-1-3 Parameter  $\beta$  (Park and Ang (1985))

The default values in STERA\_3D are

$$\mu_u = 15$$

$$\beta = 0.2$$

### 3) RC Wall

There is not much study about the damage index of RC shear walls. Therefore, STERA\_3D adopts arbitrary values for  $\mu_u$  and  $\beta$ .

STERA\_3D adopts

$$\mu_u = 15$$

$$\beta = 0.05$$

for the nonlinear flexural springs at both ends of the reinforced concrete wall.

Also

$$\mu_u = 8$$

$$\beta = 0.1$$

is adopted for the shear spring of the reinforced concrete wall.

### 4) Damage Index of group of members

The damage index for a part of a structure, such as individual story and for the entire structure, can be evaluated as the weighting average of damage indices of structural elements in the part.

$$D_{part} = \sum_i^n w_i D_i \quad (11-1-6)$$

where

$D_{part}$  : damage index of the part of the structure

$n$  : number of elements in the part of the structure

$w_i$  : weighting factor of the i-th element.

$D_i$  : damage index of the i-th element

The weighting factor  $w_i$  can be based on the dissipated hysteretic energy of each element as,

$$w_i = \frac{E_{h,i}}{\sum_i^n E_{h,i}} \quad (11-1-7)$$

## 11.2 Damage Index of Steel Members

### Reference:

- *Study on Seismic Performance for Super-High-Rise Steel Buildings against Long-Period Earthquake Ground Motions*, Building Research Institute, Building Research Data, No. 160, 2014,7 (in Japanese)

### 1) Steel Beam Connection

#### a) Damage index based on fatigue curve

The linear cumulative damage model known as the **Miner rule** is one of the frequently applied procedure to estimate the cumulative damage index (CDI) of element with random cyclic loadings. It is described as,

$$CDI = \sum_i \frac{n_i}{N_i} \leq 1 \quad (11-2-1)$$

where

$CDI$  : cumulative damage index

$n_i$  : number of cycles accumulated at strain level  $\Delta\varepsilon_i$

$N_i$  : number of cycles to fracture

For the low cycle fatigue with the cyclic plastic deformation, the relationship between the strain amplitude  $\Delta\varepsilon_i$  and the number of cycles to fracture  $N_i$  is expressed by the **Mason-Coffin equation** as,

$$\Delta\varepsilon_i (\%) = C \cdot N_i^{-\beta} \quad (11-2-2)$$

or

$$N_i = \left( \frac{\Delta\varepsilon_i}{C} \right)^{\frac{1}{\beta}} = \left( \frac{C}{\Delta\varepsilon_i} \right)^{\frac{1}{\beta}} \quad (11-2-3)$$

It can be written as follows using the ductility factor  $\mu_i$  instead of  $\Delta\varepsilon_i$ ,

$$N_i = \left( \frac{\mu_i}{C} \right)^{\frac{1}{\beta}} = \left( \frac{C}{\mu_i} \right)^{\frac{1}{\beta}} \quad (11-2-4)$$

According to Figure 11-2-1 in the report “Study on Seismic Performance for Super-High-Rise Steel Buildings against Long-Period Earthquake Ground Motions” (BRI, 2014),

$$C = 4 \sim 10$$

$$\beta = 1/3$$

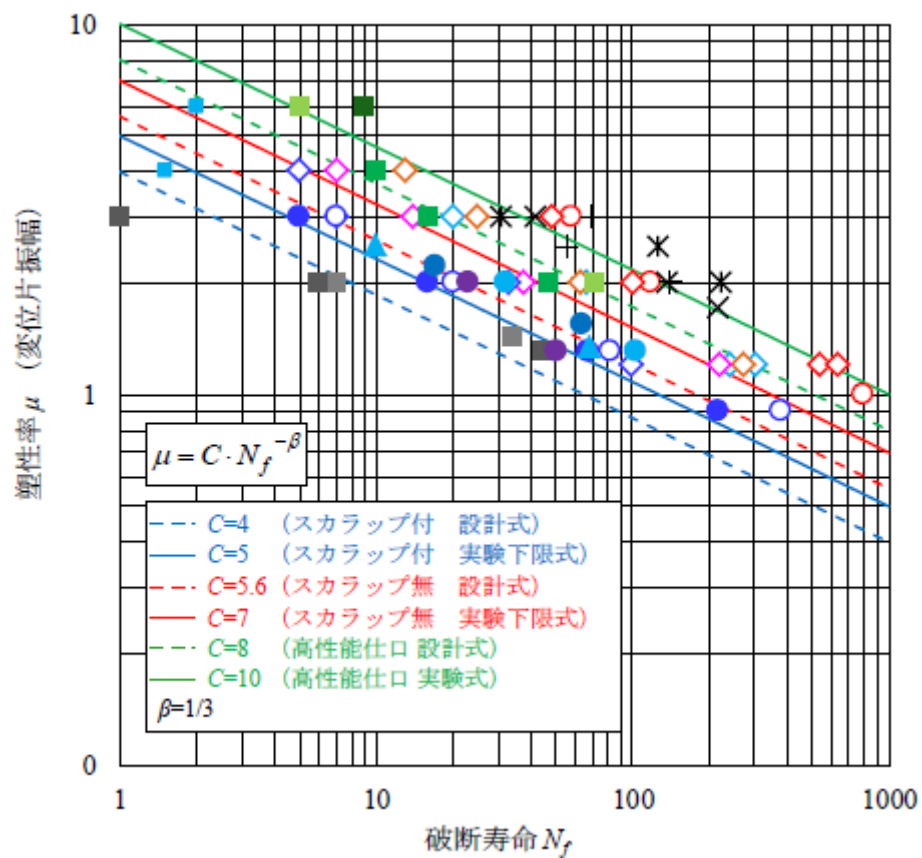


Figure 11-2-1 Fatigue curve for different connection types of steel beams (BRI, 2014)

In this method, the number of cycles  $n_i$  accumulated at strain level  $\Delta\epsilon_i$  (or ductility factor  $\mu_i$ ) must be calculated using the **Rain-flow method**.

## Appendix) Rain-flow method

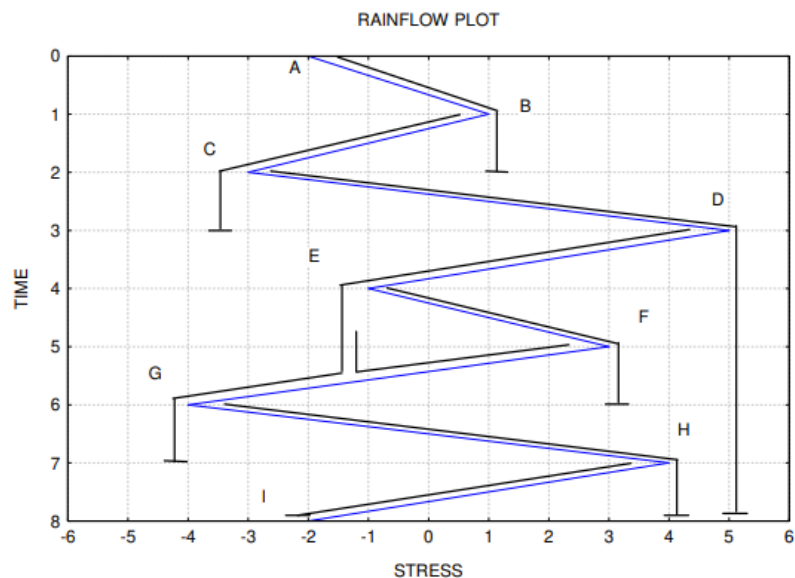
### Reference:

- RAINFLOW CYCLE COUNTING IN FATIGUE ANALYSIS, Tom Irvine, 2018

The Rain-flow algorithm is the method for counting fatigue cycles from a time history.

### Algorithm

1. Reduce the time history to a sequence of (tensile) peaks and (compressive) troughs.
2. Imagine that the time history is a pagoda.
3. Turn the sheet clockwise 90°, so the starting time is at the top.
4. Each tensile peak is imagined as a source of water that "drips" down the pagoda.
5. Count the number of half-cycles by looking for terminations in the flow occurring when either:
  - a. It reaches the end of the time history
  - b. It merges with a flow that started at an earlier tensile peak; or
  - c. It encounters a trough of greater magnitude.
6. Repeat step 5 for compressive troughs.
7. Assign a magnitude to each half-cycle equal to the stress difference between its start and termination.
8. Pair up half-cycles of identical magnitude (but opposite sense) to count the number of complete cycles. Typically, there are some residual half-cycles.

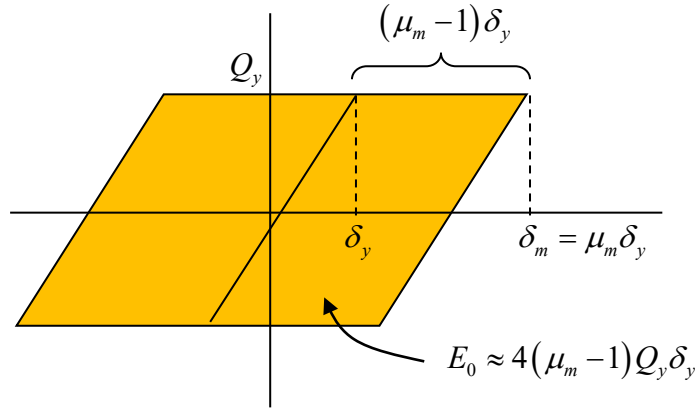


**b) Damage index based on the maximum response**

Since it is an intensive work to record the time history of strain (or ductility factor) for all beams and calculate the damage index using the rain-flow method, a practical method is proposed using the maximum ductility factor and the cumulative ductility factor (BRI, 2014).

The cumulative ductility factor is defined as

$$\eta = \frac{E_h}{Q_y \delta_y} \quad (11-2-5)$$



The energy dissipation per cycle with the deformation of the maximum ductility  $\mu_m$  is

$$E_0 \approx 4(\mu_m - 1) Q_y \delta_y \quad (11-2-6)$$

Therefore, the equivalent number of cycles is

$$N_e = \frac{E_h}{E_0} = \frac{\eta Q_y \delta_y}{4(\mu_m - 1) Q_y \delta_y} = \frac{\eta}{4(\mu_m - 1)} \quad (11-2-7)$$

The number of cycles to fracture with the maximum ductility  $\mu_m$  is

$$N_f = \left( \frac{\mu_m}{C} \right)^{-\frac{1}{\beta}} \quad (11-2-8)$$

Therefore, the damage index is evaluated as

$$CDI = \frac{N_e}{N_f} = \frac{\eta}{4(\mu_m - 1)} \left( \frac{\mu_m}{C} \right)^{\frac{1}{\beta}} \quad (11-2-9)$$

## 2) BRB (Buckling Restrained Brace)

### Reference:

- *Buckling-Restrained Braces and Applications*, Edited by T. Takeuchi and A. Wada, JSSI, 2017

### a) Damage index based on fatigue curve

The **Miner rule** is described as,

$$CDI = \sum_i \frac{n_i}{N_i} \leq 1 \quad (11-2-10)$$

The **Mason-Coffin equation** for the relationship between the strain amplitude  $\Delta\epsilon_i$  and the number of cycles to fracture  $N_i$  is expressed as,

$$\Delta\epsilon_i (\%) = C \cdot N_i^{-\beta} \quad (11-2-11)$$

For the BRB (buckling restrained brace) damper, Takeuchi et al. (2008), proposed the following formulas,

$$\begin{aligned} \Delta\epsilon_i (\%) &= 0.5 \cdot N_i^{-0.14} & (\Delta\epsilon_i (\%) < 0.1\%) \\ \Delta\epsilon_i (\%) &= 20.48 \cdot N_i^{-0.49} & (0.1\% < \Delta\epsilon_i (\%) < 2.2\%) \\ \Delta\epsilon_i (\%) &= 54.0 \cdot N_i^{-0.71} & (2.2\% < \Delta\epsilon_i (\%)) \end{aligned} \quad (11-2-12)$$

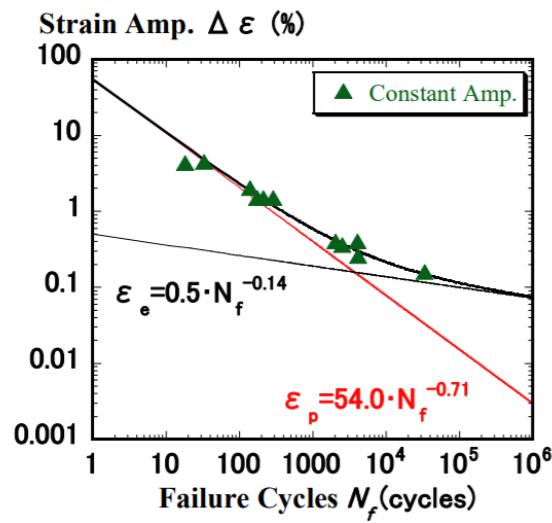


Figure 11-2-2 Relationship between strain and number of cycles to fracture (Takeuchi et al. (1985))

By combining with Coffin-Manson equations,

$$CDI = \sum \left[ \begin{array}{ccc} (\Delta \varepsilon_i < 0.1\%) & (0.1\% \leq \Delta \varepsilon_i < 2.2\%) & (2.2\% \leq \Delta \varepsilon_i) \\ \frac{n_i}{\left( \frac{\Delta \varepsilon_i}{0.5} \right)^{-\frac{1}{0.14}}} & , \quad \frac{n_i}{\left( \frac{\Delta \varepsilon_i}{20.48} \right)^{-\frac{1}{0.49}}} & , \quad \frac{n_i}{\left( \frac{\Delta \varepsilon_i}{54.0} \right)^{-\frac{1}{0.71}}} \end{array} \right] \quad (11-2-13)$$

**b) Damage index based on the maximum response**

Using the same concept as in the case of steel beams, the damage index is evaluated as

$$CDI = \frac{N_e}{N_f} = \frac{\eta}{4(\mu_m - 1)} \left( \frac{\mu_m}{C} \right)^{\frac{1}{\beta}} \quad (11-2-14)$$